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A new additive function and the F. Smarandache function

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Abstract For any positive integer n, we define the arithmetical function $F(n)$ as $F(1) = 0$. If $n > 1$ and $n = p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_k^{\alpha_k}$ be the prime power factorization of n, then $F(n) = \alpha_1 p_1 + \alpha_2 p_2 + \cdots + \alpha_k p_k$. Let $S(n)$ be the Smarandache function. The main purpose of this paper is using the elementary method and the prime distribution theory to study the mean value properties of $(F(n) - S(n))^2$, and give a sharper asymptotic formula for it.

Keywords Additive function, Smarandache function, Mean square value, Elementary method, Asymptotic formula.

§1. Introduction and result

Let $f(n)$ be an arithmetical function, we call $f(n)$ as an additive function, if for any positive integers m, n with $(m,\ n) = 1$, we have $f(mn) = f(m) + f(n)$. We call $f(n)$ as a complete additive function, if for any positive integers r and s, $f(rs) = f(r) + f(s)$. In elementary number theory, there are many arithmetical functions satisfying the additive properties. For example, if $n = p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_k^{\alpha_k}$ denotes the prime power factorization of n, then function $\Omega(n) = \alpha_1 + \alpha_2 + \cdots + \alpha_k$ and logarithmic function $f(n) = \ln n$ are two complete additive functions, $\omega(n) = k$ is an additive function, but not a complete additive function. About the properties of the additive functions, one can find them in references [1], [2] and [5].

In this paper, we define a new additive function $F(n)$ as follows: $F(1) = 0$; If $n > 1$ and $n = p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_k^{\alpha_k}$ denotes the prime power factorization of n, then $F(n) = \alpha_1 p_1 + \alpha_2 p_2 + \cdots + \alpha_k p_k$. It is clear that this function is a complete additive function. In fact if $m = p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_k^{\alpha_k}$ and $n = p_1^{\beta_1} p_2^{\beta_2} \cdots p_k^{\beta_k}$, then we have $mn = p_1^{\alpha_1+\beta_1} p_2^{\alpha_2+\beta_2} \cdots p_k^{\alpha_k+\beta_k}$. Therefore, $F(mn) = (\alpha_1 + \beta_1)p_1 + (\alpha_2 + \beta_2)p_2 + \cdots + (\alpha_k + \beta_k)p_k = F(m) + F(n)$. So $F(n)$ is a complete additive function. Now we let $S(n)$ be the Smarandache function. That is, $S(n)$ denotes the smallest positive integer m such that n divide $m!$, or $S(n) = \min\{m :\ n \mid m!\}$. About the properties of $S(n)$, many authors had studied it, and obtained a series results, see references [7], [8] and [9]. The main purpose of this paper is using the elementary method and the prime distribution theory to study the mean value properties of $(F(n) - S(n))^2$, and give a sharper asymptotic formula for it. That is, we shall prove the following:

Theorem. Let N be any fixed positive integer. Then for any real number $x > 1$, we have the asymptotic formula

$$\sum_{n\leq x}(F(n)-S(n))^2=\sum_{i=1}^{N}c_i\cdot\frac{x^2}{\ln^{i+1}x}+O\left(\frac{x^2}{\ln^{N+2}\sqrt{x}}\right),$$

where c_i $(i=1,\ 2,\ \cdots,\ N)$ are computable constants, and $c_1=\frac{\pi^2}{6}$.

§2. Proof of the theorem

In this section, we use the elementary method and the prime distribution theory to complete the proof of the theorem. We using the idea in reference [4]. First we define four sets A, B, C, D as follows: $A=\{n,\ n\in N$, n has only one prime divisor p such that $p\mid n$ and $p^2\nmid n$, $p>n^{\frac{1}{3}}\}$; $B=\{n,\ n\in N$, n has only one prime divisor p such that $p^2\mid n$ and $p>n^{\frac{1}{3}}\}$; $C=\{n,\ n\in N$, n has two deferent prime divisors p_1 and p_2 such that $p_1p_2\mid n$, $p_2>p_1>n^{\frac{1}{3}}\}$; $D=\{n,\ n\in N$, any prime divisor p of n satisfying $p\leq n^{\frac{1}{3}}\}$, where N denotes the set of all positive integers. It is clear that from the definitions of A, B, C and D we have

$$\begin{aligned}\sum_{n\leq x}(F(n)-S(n))^2 &= \sum_{\substack{n\leq x\\ n\in A}}(F(n)-S(n))^2+\sum_{\substack{n\leq x\\ n\in B}}(F(n)-S(n))^2\\ &\quad+\sum_{\substack{n\leq x\\ n\in C}}(F(n)-S(n))^2+\sum_{\substack{n\leq x\\ n\in D}}(F(n)-S(n))^2\\ &\equiv W_1+W_2+W_3+W_4. \qquad (1)\end{aligned}$$

Now we estimate W_1, W_2, W_3 and W_4 in (1) respectively. Note that $F(n)$ is a complete additive function, and if $n\in A$ with $n=pk$, then $S(n)=S(p)=p$, and any prime divisor q of k satisfying $q\leq n^{\frac{1}{3}}$, so $F(k)\leq n^{\frac{1}{3}}\ln n$. From the Prime Theorem (See Chapter 3, Theorem 2 of [3]) we know that

$$\pi(x)=\sum_{p\leq x}1=\sum_{i=1}^{k}c_i\cdot\frac{x}{\ln^i x}+O\left(\frac{x}{\ln^{k+1}x}\right), \qquad (2)$$

where c_i $(i=1,\ 2,\cdots,\ k)$ are computable constants, and $c_1=1$. By these we have the estimate:

$$\begin{aligned}W_1 &= \sum_{\substack{n\leq x\\ n\in A}}(F(n)-S(n))^2=\sum_{\substack{pk\leq x\\ (pk)\in A}}(F(pk)-p)^2\\ &= \sum_{\substack{pk\leq x\\ (pk)\in A}}F^2(k)\ll\sum_{k\leq\sqrt{x}}\sum_{k<p\leq\frac{x}{k}}(pk)^{\frac{2}{3}}\ln^2(pk)\leq(\ln x)^2\sum_{k\leq\sqrt{x}}k^{\frac{2}{3}}\sum_{k<p\leq\frac{x}{k}}p^{\frac{2}{3}}\\ &\ll (\ln x)^2\sum_{k\leq\sqrt{x}}k^{\frac{2}{3}}\left(\frac{x}{k}\right)^{\frac{5}{3}}\frac{1}{\ln\frac{x}{k}}\ll x^{\frac{5}{3}}\ln^2 x. \qquad (3)\end{aligned}$$

If $n \in B$, then $n = p^2k$, and note that $S(n) = S(p^2) = 2p$, we have the estimate

$$\begin{aligned} W_2 &= \sum_{\substack{n\le x\\ n\in B}} (F(n)-S(n))^2 = \sum_{\substack{p^2k\le x\\ p>k}} \left(F(p^2k)-2p\right)^2 \\ &= \sum_{k\le x^{\frac13}} \sum_{k<p\le\sqrt{\frac{x}{k}}} F^2(k) \ll \sum_{k\le x^{\frac13}} \sum_{k<p\le\sqrt{\frac{x}{k}}} k^2 \\ &\ll \sum_{k\le x^{\frac13}} \frac{k^2\cdot x^{\frac12}}{k^{\frac12}\ln x} \ll \frac{x^{\frac43}}{\ln x}. \end{aligned} \tag{4}$$

If $n \in D$, then $F(n) \le n^{\frac13}\ln n$ and $S(n) \le n^{\frac13}\ln n$, so we have

$$W_4 = \sum_{\substack{n\le x\\ n\in D}} (F(n)-S(n))^2 \ll \sum_{n\le x} n^{\frac23}\ln^2 n \ll x^{\frac53}\ln^2 x. \tag{5}$$

Finally, we estimate main term W_3. Note that $n \in C$, $n = p_1p_2k$, $p_2 > p_1 > n^{\frac13} > k$. If $k < p_1 < n^{\frac13}$, then in this case, the estimate is exact same as in the estimate of W_1. If $k < p_1 < p_2 < n^{\frac13}$, in this case, the estimate is exact same as in the estimate of W_4. So by (2) we have

$$\begin{aligned} W_3 &= \sum_{\substack{n\le x\\ n\in C}} (F(n)-S(n))^2 = \sum_{\substack{p_1p_2k\le x\\ p_2>p_1>k}} (F(p_1p_2k)-p_2)^2 + O\left(x^{\frac53}\ln^2 x\right) \\ &= \sum_{k\le x^{\frac13}} \sum_{k<p_1\le\sqrt{\frac{x}{k}}} \sum_{p_2\le\frac{x}{p_1k}} \left(F^2(k)+2p_1F(k)+p_1^2\right) + O\left(x^{\frac53}\ln^2 x\right) \\ &= \sum_{k\le x^{\frac13}} \sum_{k<p_1\le\sqrt{\frac{x}{k}}} \sum_{p_1<p_2\le\frac{x}{p_1k}} p_1^2 + O\left(\sum_{k\le x^{\frac13}} \sum_{k<p_1\le\sqrt{\frac{x}{k}}} \sum_{p_1<p_2\le\frac{x}{p_1k}} kp_1\right) + O\left(x^{\frac53}\ln^2 x\right) \\ &= \sum_{k\le x^{\frac13}} \sum_{k<p_1\le\sqrt{\frac{x}{k}}} p_1^2 \left(\sum_{i=1}^{N} c_i\cdot\frac{x}{p_1k\ln^i\frac{x}{p_1k}} + O\left(\frac{x}{p_1k\ln^{N+1}x}\right)\right) + O\left(x^{\frac53}\ln^2 x\right) \\ &\quad - \sum_{k\le x^{\frac13}} \sum_{k<p_1\le\sqrt{\frac{x}{k}}} p_1^2 \sum_{p_2\le p_1} 1 + O\left(\sum_{k\le x^{\frac13}} \sum_{k<p_1\le\sqrt{\frac{x}{k}}} \sum_{p_1<p_2\le\frac{x}{p_1k}} kp_1\right). \end{aligned} \tag{6}$$

Note that $\zeta(2) = \frac{\pi^2}{6}$, from the Abel's identity (See Theorem 4.2 of [6]) and (2) we have

$$\begin{aligned} &\sum_{k\le x^{\frac13}} \sum_{k<p_1\le\sqrt{\frac{x}{k}}} p_1^2 \sum_{p\le p_1} 1 = \sum_{k\le x^{\frac13}} \sum_{k<p_1\le\sqrt{\frac{x}{k}}} p_1^2 \left[\sum_{i=1}^{N} \frac{c_i\cdot p_1}{\ln^i p_1} + O\left(\frac{p_1}{\ln^{N+1} p_1}\right)\right] \\ &= \sum_{i=1}^{N} \sum_{k\le x^{\frac13}} \sum_{k<p_1\le\sqrt{\frac{x}{k}}} \frac{c_i\cdot p_1^3}{\ln^i p_1} + O\left(\sum_{k\le x^{\frac13}} \sum_{k<p_1\le\sqrt{\frac{x}{k}}} \frac{p_1^3}{\ln^{N+1} p_1}\right) \\ &= \sum_{i=1}^{N} \frac{d_i\cdot x^2}{\ln^{i+1} x} + O\left(\frac{2^N\cdot x^2}{\ln^{N+2} x}\right), \end{aligned} \tag{7}$$

where d_i $(i = 1,\ 2,\ \cdots,\ N)$ are computable constants, and $d_1 = \frac{\pi^2}{6}$.

$$\sum_{k\le x^{\frac13}}\sum_{k<p_1\le\sqrt{\frac{x}{k}}}\sum_{p_1<p_2\le\frac{x}{p_1k}} kp_1 \ll \sum_{k\le x^{\frac13}} k \sum_{p_1\le\sqrt{\frac{x}{k}}} p_1\cdot\frac{x}{p_1k\ln x} \ll \sum_{k\le x^{\frac13}}\frac{x^{\frac32}}{\sqrt{k}\ln^2 x} \ll \frac{x^{\frac53}}{\ln^2 x}. \tag{8}$$

$$\sum_{k\le x^{\frac13}}\sum_{k<p_1\le\sqrt{\frac{x}{k}}}\frac{p_1x}{k\ln^{N+1}x} \ll \sum_{k\le x^{\frac13}}\frac{x^2}{k^2\ln^{N+2}x} \ll \frac{x^2}{\ln^{N+2}x}. \tag{9}$$

From the Abel's identity and (2) we also have the estimate

$$\begin{aligned}&\sum_{k\le x^{\frac13}}\sum_{k<p_1\le\sqrt{\frac{x}{k}}} p_1^2\frac{x}{p_1k\ln\frac{x}{p_1k}} = \sum_{k\le x^{\frac13}}\frac1k\sum_{k<p_1\le\sqrt{\frac{x}{k}}}\frac{xp_1}{\ln\frac{x}{kp_1}}\\ =\ &\sum_{i=1}^{N} b_i\cdot\frac{x^2}{\ln^{i+1}x} + O\left(\frac{x^2}{\ln^{N+1}x}\right),\end{aligned} \tag{10}$$

where b_i $(i = 1,\ 2,\ \cdots,\ N)$ are computable constants, and $b_1 = \frac{\pi^2}{3}$.

Now combining (1), (3), (4), (5), (6), (7), (8)and(9) we may immediately deduce the asymptotic formula:

$$\sum_{n\le x}(F(n)-S(n))^2 = \sum_{i=1}^{N} a_i\cdot\frac{x^2}{\ln^{i+1}x} + O\left(\frac{x^2}{\ln^{N+2}\sqrt{x}}\right),$$

where a_i $(i = 1,\ 2,\ \cdots,\ N)$ are computable constants, and $a_1 = b_1 - d_1 = \frac{\pi^2}{6}$.

This completes the proof of Theorem.

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