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On the Smarandache function and the divisor product sequences

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Abstract Let n be any positive integer, $P_d(n)$ denotes the product of all positive divisors of n. The main purpose of this paper is using the elementary and analytic methods to study the mean value properties of a new arithmetical function $S(P_d(n))$, and give an interesting asymptotic formula for it.

Keywords Smarandach function, Divisor product sequences, Composite function, mean value, Asymptotic formula.

§1. Introduction

For any positive integer n, the famous F.Smarandache function $S(n)$ is defined as the smallest positive integer m such that n divide $m!$. That is, $S(n) = \min\{m : m \in N, n|m!\}$. And the Smarandache divisor product sequences $\{P_d(n)\}$ is defined as the product of all positive divisors of n. That is, $P_d(n) = \prod_{d|n} d = n^{\frac{d(n)}{2}}$, where $d(n)$ is the Dirichlet divisor function. For examples, $P_d(1) = 1, P_d(2) = 2, P_d(3) = 3, P_d(4) = 8, \cdots$. In problem 25 of reference [1], Professor F.Smarandache asked us to study the properties of the function $S(n)$ and the sequence $\{P_d(n)\}$. About these problems, many scholars had studied them, and obtained a series interesting results, see references [2], [3], [4], [5] and [6]. But at present, none had studied the mean value properties of the composite function $S(P_d(n))$, at least we have not seen any related papers before. In this paper, we shall use the elementary methods to study the mean value properties of $S(P_d(n))$, and give an interesting asymptotic formula for it. That is, we shall prove the following conclusion:

Theorem. For any fixed positive integer k and any real number $x \geq 1$, we have the asymptotic formula

$$\sum_{n\leq x} S(P_d(n)) = \frac{\pi^4}{72} \cdot \frac{x^2}{\ln x} + \sum_{i=2}^{k} b_i \cdot \frac{x^2}{\ln^i x} + O\left(\frac{x^2}{\ln^{k+1} x}\right),$$

where b_i $(i = 2, 3, \cdots, k)$ are computable constants.

§2. Some simple lemmas

To complete the proof of the theorem, we need the following several simple lemmas. First we have

Lemma 1. For any positive integer α, we have the estimate

$$S(p^{\alpha}) \leq \alpha p.$$

Especially, when $\alpha \leq p$, we have $S(p^{\alpha}) = \alpha p$, where p is a prime.

Proof. See reference [3].

Lemma 2. For any positive integer n, let $n = p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_k^{\alpha_k}$ denotes the factorization of n into prime powers, then we have

$$S(n) = \max_{1\leq i\leq k} \{S(p_i^{\alpha_i})\}.$$

Lemma 3. Let $P(n)$ denotes the greatest prime divisor of n, if $P(n) > \sqrt{n}$, then we have $S(n) = P(n)$.

Proof. The proof of Lemma 2 and Lemma 3 can be found in reference [4].

§3. Proof of the theorem

In this section , we shall use the above lemmas to complete the proof of our theorem. For any positive integer n, it is clear that from the definition of $P_d(n)$ we have

$$P_d^2(n) = \left(\prod_{r|n} r\right) \cdot \left(\prod_{r|n} \frac{n}{r}\right) = n^{\sum\limits_{r|n} 1} = n^{d(n)}.$$

So we have the identity $P_d(n) = n^{\frac{d(n)}{2}}$. Let $n = p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_k^{\alpha_k}$ denotes the factorization of n into prime powers. First we separate all integers n in the interval $[1,\ x]$ into two subsets A and B as follows:

$$A = \{n:\ n \leq x,\ P(n) \leq \sqrt{n}\}, \quad B = \{n:\ n \leq x,\ P(n) > \sqrt{n}\}.$$

If $n \in A$, then from Lemma 1 and Lemma 2, and note that $P_d(n) = n^{\frac{d(n)}{2}}$ we have

$$P_d(n) = n^{\frac{d(n)}{2}} = p_1^{\frac{\alpha_1 d(n)}{2}} p_2^{\frac{\alpha_2 d(n)}{2}} \cdots p_k^{\frac{\alpha_k d(n)}{2}}.$$

Therefore,

$$\begin{aligned} S\left(P_d(n)\right) &= S\left(p_1^{\frac{\alpha_1 d(n)}{2}} p_2^{\frac{\alpha_2 d(n)}{2}} \cdots p_k^{\frac{\alpha_k d(n)}{2}}\right) = \max_{1\leq i\leq k}\left\{S\left(p_i^{\frac{\alpha_i d(n)}{2}}\right)\right\} \\ &\leq \max_{1\leq i\leq k}\left\{\frac{\alpha_i d(n)}{2} p_i\right\} \leq \frac{d(n)}{2}\sqrt{n}\ln n. \end{aligned}$$

From reference [10] we know that

$$\sum_{n\leq x} d(n) = x\ln x + O(x).$$

So we have the estimate

$$\sum_{n\in A} S\left(P_d(n)\right) \leq \sum_{n\in A} \frac{d(n)}{2}\sqrt{n}\ln n \ll \sum_{n\leq x} d(n)\sqrt{x}\ln x \ll x^{\frac{3}{2}}\ln^2 x. \tag{1}$$

If $n \in B$, let $n = n_1 p$, where $n_1 < \sqrt{n} < p$. It is clear that $d(n_1) < \sqrt{n} < p$ and $d(n) = 2d(n_1)$. So from Lemma 3 we have

$$\begin{aligned}
\sum_{n\in B} S\left(P_d(n)\right) &= \sum_{\substack{n_1p\leq x\\ n_1<p}} S\left((n_1p)^{\frac{d(n_1p)}{2}}\right) = \sum_{\substack{n_1p\leq x\\ n_1<p}} S\left(p^{\frac{d(n_1p)}{2}}\right)\\
&= \sum_{n\leq\sqrt{x}}\sum_{n<p\leq\frac{x}{n}} d(n)p = \sum_{n\leq\sqrt{x}} d(n) \sum_{n<p\leq\frac{x}{n}} p\\
&= \sum_{n\leq\sqrt{x}} d(n) \sum_{p\leq\frac{x}{n}} p + O\left(\sum_{n\leq\sqrt{x}} d(n)\cdot\frac{n}{\ln n}\right)\\
&= \sum_{n\leq\sqrt{x}} d(n) \sum_{p\leq\frac{x}{n}} p + O(x).
\end{aligned} \tag{2}$$

From the Abel's summation formula (see Theorem 4.2 of [10]) and the Prime Theorem (see Theorem 3.2 of [11]) we have

$$\pi(x) = \sum_{i=1}^{k} \frac{a_i\cdot x}{\ln^i x} + O\left(\frac{x}{\ln^{k+1} x}\right),$$

where a_i $(i = 1,\ 2,\ \cdots,\ k)$ are computable constants and $a_1 = 1$. We have

$$\begin{aligned}
\sum_{p\leq\frac{x}{n}} p &= \frac{x}{n}\pi\left(\frac{x}{n}\right) - \int_2^{\frac{x}{n}} \pi(y)dy\\
&= \frac{x^2}{2n^2\ln x} + \sum_{i=2}^{k} \frac{c_i\cdot x^2\ln^i n}{n^2\ln^2 x} + O\left(\frac{x^2}{n^2\ln^{k+1} x}\right),
\end{aligned} \tag{3}$$

where c_i $(i = 2,\ 3,\ \cdots,\ k)$ are computable constants.

Note that

$$\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}$$

and

$$\sum_{n=1}^{\infty} \frac{d(n)}{n^2} = \left(\sum_{n=1}^{\infty}\frac{1}{n^2}\right)^2 = \frac{\pi^4}{36}, \tag{4}$$

from (2), (3) and (4) we obtain

$$\begin{aligned}\sum_{n\in B} S\left(P_d(n)\right) &= \frac{x^2}{2\ln x}\sum_{n\le\sqrt{x}}\frac{d(n)}{n^2}+\sum_{n\le\sqrt{x}}\sum_{i=2}^{k}\frac{c_i\cdot x^2 d(n)\ln^i n}{n^2\ln^i x}+O\left(\frac{x^2}{\ln^{k+1}x}\right)\\ &= \frac{\pi^4}{72}\frac{x^2}{\ln x}+\sum_{i=2}^{k} b_i\cdot\frac{x^2}{\ln^i x}+O\left(\frac{x^2}{\ln^{k+1}x}\right), \qquad (5)\end{aligned}$$

where b_i $(i=2,\ 3,\ \cdots,\ k)$ are computable constants.

Now combining (1) and (5) we may immediately get the asymptotic formula

$$\begin{aligned}\sum_{n\le x} S\left(P_d(n)\right) &= \sum_{n\in A} S\left(P_d(n)\right)+\sum_{n\in B} S\left(P_d(n)\right)\\ &= \frac{\pi^4}{72}\cdot\frac{x^2}{\ln x}+\sum_{i=2}^{k} b_i\cdot\frac{x^2}{\ln^i x}+O\left(\frac{x^2}{\ln^{k+1}x}\right),\end{aligned}$$

where b_i $(i=2,\ 3,\ \cdots,\ k)$ are computable constants. This completes the proof of Theorem.

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