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A survey on Smarandache notions in number theory VII: Smarandache multiplicative function

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Abstract In this paper we give a survey on recent results on Smarandache multiplicative function.

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§1. Definition and the mean value properties of the Smarandache multiplicative function

For any positive integer n, $f(n)$ is called a Smarandache multiplicative function if $f(ab) = \max(f(a), f(b)), (a, b) = 1$, and if $n = p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_k^{\alpha_k}$ is the prime powers factorization of n, then

$$f(n) = \max_{1 \le i \le k} \{f(p_i^{\alpha_i})\}, \tag{1.1}$$

for any prime p and any positive integer α, $f(n)$ is a new Smarandache multiplicative function if $f(p^\alpha) = \alpha p$. That is

$$f(n) = \max_{1 \le i \le k} \{f(p_i^{\alpha_i})\} = \max_{1 \le i \le k} \{\alpha_i p_i\}.$$

J. Ma [11]. *For any real number $x \ge 2$, we have the asymptotic formula*

$$\sum_{n \le x} f(n) = \frac{\pi^2}{12} \cdot \frac{x^2}{\ln x} + O\left(\frac{x^2}{\ln^2 x}\right).$$

Y. Liu, P. Gao [10]. *A new arithmetical function $P_d(n)$ is defined as*

$$P_d(n) = \prod_{d|n} d = n^{\frac{d(n)}{2}},$$

where $d(n) = \sum_{d|n} 1$ is the Dirichlet divisor function. For any real number $x \ge 2$, we have the asymptotic formula

$$\sum_{n \le x} f(P_d(n)) = \frac{\pi^4}{72} \cdot \frac{x^2}{\ln x} + C \cdot \frac{x^2}{\ln^2 x} + O\left(\frac{x^2}{\ln^3 x}\right),$$

where $C = \frac{5\pi^4}{288} + \frac{1}{2} \sum_{n=1}^{\infty} \frac{d(n) \ln n}{n^2}$ is a constant.

X. Zhang [24]. *For any integer $n \in \mathbb{N}^+$, n is named as a simple number if the product of all proper divisors of n is no more than n. Now let A be a simple number set, that is $A = \{2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 13, 14, 15, 17, 19, 21, \ldots\}$. For any real number $x \geq 2$ we have the asymptotic formula*

$$\sum_{\substack{n \leq x \\ n \in A}} f(n) = D_1 \frac{x^2}{\ln x} + D_2 \frac{x^2}{\ln^2 x} + \frac{2x}{\ln x} + \frac{9x^{2/3}}{2 \ln x} + O\left(\frac{x^2}{\ln^3 x}\right),$$

where D_1, D_2 are computable constants.

W. Xiong [19]. *Let $OF(N)$ denotes the number of all integers $1 \leq k \leq n$ such that $f(n)$ is odd, $EF(n)$ denotes the number of all integer $1 \leq k \leq n$ such that $f(n)$ is even. For any positive integer $n > 1$, we have the asymptotic formula*

$$\frac{EF(n)}{OF(n)} = O\left(\frac{1}{\ln n}\right).$$

From the formula above, it can be immediately deduced the following

$$\lim_{n \to \infty} \frac{EF(n)}{OF(n)} = 0.$$

J. Li [6]. *For any real number $x > 1$, we have the asymptotic formula*

$$\sum_{\substack{n \in \mathbb{N} \\ f(n) \leq x}} = \mathrm{e}^{c \frac{x}{\ln x} + O\left(\frac{x (\ln \ln x)^2}{\ln^2 x}\right)},$$

where $c = \sum_{n=1}^{\infty} \frac{\ln(n+1)}{n(n+1)}$ is a constant.

Z. Feng [1]. *A natural number n is of the k-full number if for any prime p, $p \mid n$ implies $p^k \mid n$. Let A_k be a simple number set, for any real number $x \geq 2$ we have the asymptotic formula*

$$\sum_{\substack{n \leq x \\ n \in A_k}} f(n) = C_1 \frac{x^2}{\ln x} + C_2 \frac{x^2}{\ln^2 x} + \frac{2x}{\ln x} + \frac{9x^{2/3}}{2 \ln x} + O\left(\frac{x^2}{\ln^3 x}\right),$$

where C_1, C_2 are computable constants.

Y. Men [12]. *Let $Smd(n) = \sum_{d \mid n} \frac{1}{f(d)}$, for any real number $x \geq 1$, when $n \neq 1, 24$, we have*

(1). *If $n = p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_s^{\alpha_s} p$, $p_1^{\alpha_1} < p_2^{\alpha_2} < \cdots < p_s^{\alpha_s} < p$, and p, $p_i (i = 1, 2, \ldots, s)$ are primes, then $Smd(n)$ is not a positive integer;*

(2). *If $n = p_1 p_2 \cdots p_s, p_1 < p_2 < \cdots < p_s$, $p_i (i = 1, 2, \ldots, s)$ are primes, then $Smd(n)$ is not a positive integer.*

R. Guo and X. Zhao [2]. 1. *For any real number $x \geq 1$ and any fixed positive integer $k \geq 2$, we have the asymptotic formula*

$$\sum_{n \leq x} \Lambda(n) f(n) = x^2 \sum_{i=1}^{k} \frac{c_i}{\ln^{i-1} x} + O\left(\frac{x^2}{\ln^k x}\right),$$

where $\Lambda(n)$ *is the Mangoldt function,* $c_i (i = 1, 2, \ldots, k)$ *are computable constants and* $c_1 = \frac{1}{2}$.

2. *For any real number* $x \geq 1$ *and any fixed positive integer* $k \geq 2$*, we have the asymptotic formula*

$$\sum_{n \leq x} \Lambda(n) S(n) = x^2 \sum_{i=1}^{k} \frac{c_i}{\ln^{i-1} x} + O\left(\frac{x^2}{\ln^k x}\right),$$

where $S(n)$ *is the famous Smarandache function,* $S(n) = \min\{m : m \in \mathbb{N}, n \mid m!\}$*,* $c_i (i = 1, 2, \ldots, k)$ *are computable constants and* $c_1 = \frac{1}{2}$.

For any positive integers m and n, an arithmetical function $h(n)$ is defined as follows

$$(m, n) = 1 \Rightarrow h(mn) = \max\{h(m), h(n)\}.$$

If $n = p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_k^{\alpha_k}$ is the prime powers factorization of n, defining

$$h(1) = 1, h(n) = \max_{1 \leq i \leq k} \{\frac{1}{\alpha_i + 1}\}, \tag{1.2}$$

then $h(n)$ is also a Smarandache multiplicative function.

J. Zhang and P. Zhang [22]. 1. *For any real number* $x > 1$*, we have the asymptotic formula*

$$\sum_{n \leq x} h(n) = \frac{1}{2} \cdot x + O(x^{\frac{1}{2}}).$$

2. *For any real number* $x > 1$*, we have the asymptotic formula*

$$\sum_{n \leq x} \left(h(n) - \frac{1}{2}\right)^2 = \frac{1}{36} \cdot \frac{\zeta(\frac{3}{2})}{\zeta(3)} \cdot \sqrt{x} + O(x^{\frac{1}{3}}),$$

where $\zeta(n)$ *is the Riemann Zeta-function.*

The Smarandache multiplicative function $g(n)$ can also be defined as follows

$$g(1) = 0, (m, n) = 1 \Rightarrow g(mn) = \min\{g(m), g(n)\}. \tag{1.3}$$

If $n = p_1^{t_1} p_2^{t_2} \cdots p_r^{t_r}$ is the prime powers factorization of n, then

$$g(n) = \min_{1 \leq i \leq r} \{f(p_i^{t_i})\}, \tag{1.4}$$

specifically let $g(p^t) = \min\{t, p\}$, then $g(n)$ is a new Smarandache multiplicative function.

Z. Ren [13]. *For any real number* $x > 1$*, we have the asymptotic formula*

$$\sum_{n \leq x} g(n) = x + \frac{12x^{1/2}}{\pi^2} \prod_p \left(1 + \frac{1}{(p+1)(p^{\frac{1}{2}} - 1)}\right) + \frac{18x^{1/3}}{\pi^2} \prod_p \left(1 + \frac{1}{(p+1)(p^{\frac{1}{3}} - 1)}\right) + O(x^{\frac{1}{4} + X}),$$

where X *is any fixed positive number.*

L. Li [8]. 1. *For any positive integer* n*, if* $n = p_1^{t_1} p_2^{t_2} \cdots p_r^{t_r}$ *is the prime powers factorization of* n*, let* $\lambda = \max_{1 \leq i \leq r} \{t_i\}, i = 1, \ldots, r$ *and*

$$F(1) = 1, F(n) = \min_{1 \leq i \leq r} \{\frac{1}{t_i + 1}\} = \frac{1}{\lambda + 1}, \tag{1.5}$$

then $F(n)$ is a Smarandache multiplicative function. For any real number $x > 1$, we have the asymptotic formula

$$\sum_{n\leq x} F(n) = \frac{1}{\lambda+1}x + O(x^{\frac{1}{2}}).$$

2. *For any real number $x > 1$, we have the asymptotic formula*

$$\sum_{n\leq x}\left(F(n)-\frac{1}{2}\right)^2 = \frac{12}{\pi^2}\sqrt{x} + O(x^{\frac{1}{3}}).$$

T. Zhang [23]. *Let p be a prime and for any positive real number m, $U_m(n)$ is defined as follows*

$$U(1) = 1, U_m(p^\alpha) = p^\alpha + m, \tag{1.6}$$

if $n = p_1^{\alpha_1}p_2^{\alpha_2}\cdots p_k^{\alpha_k}$ is the prime powers factorization of n, $U_m(n)$ is defined as $U_m(n) = U_m(p_1^{\alpha_1})\cdots U_m(p_k^{\alpha_k})$. For any real number $x > 1$, we have the asymptotic formula

$$\sum_{n\leq x} U_m(n) = \frac{1}{2}x^2\prod_p\left(1+\frac{m}{p(p+1)}\right) + O(x^{\frac{3}{2}+\varepsilon}).$$

X. Wang [18]. *Let $I(n)$ be the multiplicative function such that for any prime p and any integer $\alpha \geq 1$, one has*

$$I(p^\alpha) = \frac{p^{\alpha+1}}{\alpha+1},$$

then we have

$$\sum_{mn\leq x} I(m)I(n) = Cx^3 + O(x^{\frac{5}{2}+\varepsilon}),$$

where C is an explicit constant.

L. Wang [16]. *Let $N_0 \geq 1$ be a fixed integer and for the multiplicative function $I(n)$, we have*

$$\sum_{n\leq x} I(n) = x^3\log^{\frac{1}{2}} x\left(\sum_{i=1}^{N_0} c_i\log^{-i} x + O(\log^{-N_0-1} x)\right),$$

where $c_i (i \geq 1)$ are computable constants.

§2. Some hybrid mean values involving the Smarandache multiplicative function

Y. Yi [21]. *For any prime p and positive integer α, the Smarandache multiplicative function $f(n)$ is defined as $f(p^\alpha) = p^{\frac{1}{\alpha}}$. Let $n = p_1^{\alpha_1}p_2^{\alpha_2}\cdots p_r^{\alpha_r}$ is the prime powers factorization of n, then from the definition of $f(p^\alpha)$ we have*

$$f(n) = \max_{1\leq i\leq r}\{f(p_i^{\alpha_i})\} = \max_{1\leq i\leq r}\left\{p_i^{\frac{1}{\alpha_i}}\right\}.$$

For any real number $x \geq 3$, we have the asymptotic formula

$$\sum_{n \leq x} (f(n) - P(n))^2 = \frac{2\zeta(\frac{3}{2})x^{\frac{3}{2}}}{3\ln x} + O\left(\frac{x^{\frac{3}{2}}}{\ln^2 x}\right),$$

where $\zeta(n)$ denotes the Riemann zeta-function and $P(n)$ is the greatest prime divisor of n.

W. Lu and L. Gao [9]. *For any real number $x \geq 3$ and any real number or complex number α, we have the asymptotic formula*

$$\sum_{n \leq x} \delta_\alpha(n) (f(n) - P(n))^2 = \frac{\zeta(\alpha+3)\zeta(2\alpha+3)x^{2\alpha+3}}{(2\alpha+3)\ln x} + \sum_{i=2}^{r} \frac{c_i \cdot x^{2\alpha+3}}{\ln^i x} + O\left(\frac{x^{2\alpha+3}}{\ln^{r+1} x}\right),$$

where $\zeta(n)$ denotes the Riemann zeta-function and all c_i are computable constants.

H. Shen [14]. *For any positive integer n, if $n = p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_r^{\alpha_r}$ is the prime powers factorization of n, the Smarandache multiplicative function $V(n)$ is defined as follows*

$$V(1) = 1, V(n) = \max_{1 \leq i \leq r} \{\alpha_1 p_1, \ldots, \alpha_r p_r\}. \tag{2.1}$$

For any real number $x \geq 1$ and any fixed positive integer r, we have the asymptotic formula

$$\sum_{n \leq x} (V(n) - p(n))^2 = x^{\frac{3}{2}} \sum_{i=1}^{r} \frac{c_i}{\ln^i x} + O\left(\frac{x^{\frac{3}{2}}}{\ln^{r+1} x}\right),$$

where $p(n)$ is the least prime divisor of n and all c_i are computable constants.

H. Liu and W. Cui [3]. *Let $n \geq 1$ is a positive integer, we have the asymptotic formula*

$$\sum_{n \leq x} V(n)p(n) = \sum_{i=1}^{r} \frac{x^3 a_i}{\ln^i x} + O\left(\frac{x^3}{\ln^{r+1} x}\right),$$

where all $a_i (i = 1, \ldots, r)$ are computable constants.

§3. Mean values involving the Smarandache-type multiplicative function

The Smarandache-type multiplicative function $C_m(n)$ is defined as the m-th root of the largest m-th power dividing n, $J_m(n)$ is denoted as m-th root of the smallest m-th power divisible by n.

H. Liu and J. Gao [5]. 1. *For any integer $m \geq 3$ and real number $x \geq 1$, we have*

$$\sum_{n \leq x} C_m(n) = \frac{\zeta(m-1)}{\zeta(m)} x + O\left(x^{\frac{1}{2}+\epsilon}\right).$$

2. *For any integer $m \geq 1$ and real number $x \geq 1$, we have*

$$\sum_{n \leq x} J_m(n) = \frac{x^2}{2\zeta(2)} \prod_p \left[1 + \frac{\frac{1}{p^{2m}} + \frac{1}{p^3} - \frac{1}{p^{2m+1}} - \frac{1}{p^{2m+2}}}{(1+\frac{1}{p})(1-\frac{1}{p^2})(1-\frac{1}{p^{2m-1}})}\right] + O(x^{\frac{3}{2}+\epsilon}).$$

H. Liu and J. Gao [4]. 1. *For any integer $m \geq 3$ and real number $x \geq 1$, we have*

$$\sum_{n \leq x} \Lambda(n) C_m(n) = x + O\left(\frac{x}{\log x}\right),$$

where $\Lambda(n)$ *is the Mangoldent function.*

2. *For any integer* $m \geq 2$ *and real number* $x \geq 1$, *we have*

$$\sum_{n \leq x} \Lambda(n) J_m(n) = x^2 + O\left(\frac{x^2}{\log x}\right),$$

The Smarandache-type multiplicative function $K_m(n)$ is the largest m-th power-free number dividing n, $L_m(n)$ is denoted as: n divided by the largest m-th power-free number dividing n. That is, if $n = p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_k^{\alpha_k}$ is the prime powers factorization of n, it follows that

$$K_m(n) = p_1^{\beta_1} p_2^{\beta_2} \cdots p_k^{\beta_k},$$
$$L_m(n) = p_1^{\gamma_1} p_2^{\gamma_2} \cdots p_k^{\gamma_k},$$

where $\beta_i = \min(\alpha_i, m-1)$, $\gamma_i = \max(0, \alpha_i - m + 1)$

C. Yang and C. Li [20]. 1. *Let* $m \geq 2$ *is a given integer, then for any real number* $x \geq 1$, *we have*

$$\sum_{n \leq x} K_m(n) = \frac{x^2}{2\zeta(m)} \prod_p \left(1 + \frac{1}{(p^m - 1)(p+1)}\right) + O\left(x^{\frac{3}{2}+\epsilon}\right).$$

2. *Let* $m \geq 2$ *is a given integer, then for any real number* $x \geq 1$, *we have*

$$\sum_{n \leq x} \frac{1}{L_m(n)} = \frac{x}{\zeta(m)} \prod_p \left(1 + \frac{1}{(p^m - 1)(p+1)}\right) + O\left(x^{\frac{1}{2}+\epsilon}\right),$$

where $\zeta(s)$ *is the Riemann Zeta-function.*

J. Wang [15]. *The asymptotic formula*

$$\sum_{n \leq x} K_m(n) = \frac{x^2}{2\zeta(m)} \prod_p \left(1 + \frac{1}{(p^m - 1)(p+1)}\right) + O\left(x^{1+\frac{1}{m}} e^{-c_0 \delta(x)}\right).$$

holds, where c_0 *is an absolute positive constant and* $\delta(x) = (\log x)^{3/5} (\log \log x)^{-1/5}$.

For any fixed positive integer n with the normal factorization $p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_k^{\alpha_k}$, $(1 \leq i \leq k)$, the Smarandache-type multiplicative function $F_m(n)$, $G_m(n)$ are denoted as

$$F_m(p_i^{\alpha_i}) = \begin{cases} 1, & \text{if } \alpha_i = mk, \\ p_i^m, & \text{otherwise} \end{cases}.$$

and

$$G_m(p_i^{\alpha_i}) = \begin{cases} 1, & \text{if } \alpha_i = mk, \\ p_i, & \text{otherwise} \end{cases}.$$

J. Li and D. Liu [7]. 1. *For any integer* $m \geq 2$ *and real number* $x \geq 1$, *we have*

$$\sum_{n \leq x} F_m(n) = \frac{6\zeta(m^2 + m)\zeta(m+1)R(m+1)x^{m+1}}{\pi^2} + O\left(x^{m+\frac{1}{2}+\epsilon}\right),$$

where ϵ be any fixed positive integer,and

$$R(m+1) = \prod_p \left(1 - \frac{1}{p^{m+1}+p^m} - \frac{1}{p^{m^2}+p^{m^2-1}}\right).$$

2. *For any integer $m \geq 2$ and real number $x \geq 1$, we have*

$$\sum_{n\leq x} G_m(n) = \zeta(2m)R(2)x^2 + O(x^{\frac{3}{2}+\epsilon}),$$

where

$$R(2) = \prod_p \left(1 - \frac{1}{p^2+p} - \frac{1}{p^{2m-1}+p^{2m-2}}\right).$$

M. Wang [17]. 1. *For any integer $m \geq 2$, A be a set without m-th power factor number, we have*

$$\sum_{\substack{n\leq x\\ n\in A}} F_m(n) = \frac{6\zeta(m+1)x^{m+1}}{\pi^2} \prod_p \left(1 - \frac{1}{p^{m-1}+p^m} - \frac{1}{p^{m^2}+p^{m^2-1}}\right) + O\left(x^{m+\frac{1}{2}-\epsilon}\right),$$

where ϵ be any fixed positive number.

2. *For any positive integer $m \geq 2$, A be a set without m-th power factor number, we have*

$$\sum_{\substack{n\leq x\\ n\in A}} G_m(n) = x^2 \prod_p \left(1 - \frac{1}{p^2+p^m} - \frac{1}{p^{2m-1}+p^{2m-2}}\right) + O\left(x^{\frac{3}{2}-\epsilon}\right).$$

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