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Approximation To The Smarandache Curves in the The Null Cone

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Abstract In this paper, we study the Smarandache curves according to the asymptotic orthonormal frame in Null Cone $\mathbf{Q}^3$. By using cone frame formulas, we obtain some characterizations of the Smarandache curves and introduce cone frenet invariants of these curves.

1 Introduction

The idea of studying curves has been one of the impressive topic owing to having many application area from mathematics to the diverse branch of science. As a result of this case, many mathematicians have studied different type of curves by using Frenet frame in numerious spaces. Among these, Smarandache curves have attract major attention by investigators for a long while.

Smarandache geometry is a geometry which has at least one Smarandachely denied axiom [4]. An axiom is said to be Smarandachely denied, if it behaves in at least two different ways within the same space. Smarandache curve is defined as a regular curve whose position vector is composed by Frenet frame vectors of another regular curve. Smarandache curves in various ambient spaces have been classfied in [1]-[8], [14]-[16].

In this study, we give special Smarandache curves such as $x\alpha, xy, x\beta, \alpha\beta, y\beta, \alpha y$ -smarandache curves according to asymptotic orthonormal frame in the Null Cone $\mathbf{Q}^3$ and we examine the curvature and the asymptotic orthonormal frame's vectors of the Smarandache curves. We also present an example related to these curves.

2 Preliminaries

Some basics of the curves in the null cone are provided from, [9]- [10]. Let E_1^4 be the 4-dimensional pseudo-Euclidean space with the

$$\widetilde{g}(X,Y) = \langle X,Y\rangle = x_1y_1 + x_2y_2 + x_3y_3 - x_4y_4$$

for all $X = (x_1, x_2, x_3, x_4), Y = (y_1, y_2, y_3, y_4) \in E_1^4$. E_1^4 is a flat pseudo-Riemannian manifold of signature $(3, 1)$.

Let M be a submanifold of E_1^4. If the pseudo-Riemannian metric $\widetilde{g}$ of E_1^4 induces a pseudo-Riemannian metric g(respectively, a Riemannian metric, a degenerate quadratic form) on M, then M is called a timelike(respectively, spacelike, degenerate) submanifold of E_1^3. Let c be a fixed point in E_1^4. The pseudo-Riemannian lightlike cone(quadric cone) is defined by

$$\mathbf{Q}_1^3(c) = \left\{x \in E_1^4 : g(x-c, x-c) = 0\right\},$$

where the point c is called the center of $\mathbf{Q}_1^3(c)$. When $c = 0$, we simply denote $\mathbf{Q}_1^3(0)$ by $\mathbf{Q}^3$and call it the null cone.

Let E_1^4 be 4-dimensional Minkowski space and $\mathbf{Q}^3$ be the lightlike cone in E_1^4. A vector $V \neq 0$ in E_1^4 is called spacelike, timelike or lightlike, if $\langle V,V\rangle > 0$, $\langle V,V\rangle < 0$ or $\langle V,V\rangle = 0$, respectively. The norm of a vector $x \in E_1^4$ is given by $\|x\| = \sqrt{\langle x,x\rangle}$, [13].

We assume that curve $x : I \to \mathbf{Q}^3 \subset E_1^4$ is a regular curve in $\mathbf{Q}^3$ for $t \in I$. In the following, we always assume that the curve is regular.

A frame field $\{x, \alpha, \beta, y\}$ on E_1^4 is called an asymptotic orthonormal frame field, if

$$\begin{aligned} \langle x, x\rangle &= \langle y, y\rangle = \langle x, \alpha\rangle = \langle y, \alpha\rangle = \langle \beta, \alpha\rangle = \langle y, \beta\rangle = \langle x, \beta\rangle = 0, \\ \langle x, y\rangle &= \langle \alpha, \alpha\rangle = \langle \beta, \beta\rangle = 1. \end{aligned}$$

Using $x'(s) = \alpha(s)$, we know that $\{x(s), \alpha(s), \beta(s), y(s)\}$ from an asymptotic orthonormal frame along the curve $x(s)$ and the cone frenet formulas of $x(s)$ are given by

$$\begin{aligned} x'(s) &= \alpha(s) \\ \alpha'(s) &= \kappa(s)x(s) - y(s) \\ \beta'(s) &= \tau(s)x(s) \\ y'(s) &= -\kappa(s)\alpha(s) - \tau(s)\beta(s), \end{aligned} \tag{2.1}$$

where the functions $\kappa(s)$ and $\tau(s)$ are called cone curvature functions of the curve $x(s)$, [11].

Let $x : I \to \mathbf{Q}^3 \subset E_1^4$ be a spacelike curve in $\mathbf{Q}^3$ with an arc length parameter s. Then $x = x(s) = (x_1, x_2, x_3, x_4)$ can be written as

$$x(s) = \frac{1}{2\sqrt{f_s^2 + g_s^2}}(2f, 2g, 1 - f^2 - g^2, 1 + f^2 + g^2), \tag{2.2}$$

for some non constant function $f(s)$ and $g(s)$, [12].

3 The Smarandache Curves in The Null Cone Q^3

In this section, we define binary Smarandache curves according to the asymptotic orthonormal frame in $\mathbf{Q}^3$. Also, we obtain the asymptotic orthonormal frame and cone curvature functions of the Smarandache partners lying on $\mathbf{Q}^3$ using cone frenet formulas.

Smarandache curve $\gamma = \gamma(s^*(s))$ of the curve x is a regular unit speed curve lying fully on $\mathbf{Q}^3$. Let $\{x, \alpha, \beta, y\}$ and $\{\gamma, \alpha_\gamma, \beta_\gamma, y_\gamma\}$ be the moving asymptotic orthonormal frames of x and γ, respectively.

Definition 3.1. Let x be unit speed spacelike curve lying on $\mathbf{Q}^3$ with the moving asymptotic orthonormal frame $\{x, \alpha, \beta, y\}$. Then, $x\alpha$-smarandache curve of x is defined by

$$\gamma_{x\alpha}(s^*) = \frac{a}{b}x(s) + \alpha(s), \tag{3.1}$$

where $a, b \in \mathbb{R}_0^+$.

Theorem 3.2. *Let x be unit speed spacelike curve in $\mathbf{Q}^3$ with the moving asymptotic orthonormal frame $\{x, \alpha, \beta, y\}$ and cone curvatures $\kappa(s), \tau(s)$ and let $\gamma_{x\alpha}$ be $x\alpha$-smarandache curve with asymptotic orthonormal frame $\{\gamma_{x\alpha}, \alpha_{x\alpha}, \beta_{x\alpha}, y_{x\alpha}\}$. Then the following relations hold:*

i) *The asymptotic orthonormal frame $\{\gamma_{x\alpha}, \alpha_{x\alpha}, \beta_{x\alpha}, y_{x\alpha}\}$ of the $x\alpha$-smarandache curve $\gamma_{x\alpha}$ is given as*

$$\begin{bmatrix} \gamma_{x\alpha} \\ \alpha_{x\alpha} \\ \beta_{x\alpha} \\ y_{x\alpha} \end{bmatrix} = \begin{bmatrix} \frac{a}{b} & 1 & 0 & 0 \\ \frac{b\kappa}{\sqrt{a^2-2b^2\kappa}} & \frac{a}{\sqrt{a^2-2b^2\kappa}} & 0 & \frac{-b}{\sqrt{a^2-2b^2\kappa}} \\ B_1 & B_2 & B_3 & B_4 \\ \Upsilon_1 & \Upsilon_2 & \Upsilon_3 & \Upsilon_4 \end{bmatrix} \begin{bmatrix} x \\ \alpha \\ \beta \\ y \end{bmatrix}, \tag{3.2}$$

where

$$\xi = \frac{1}{\sqrt{a^2 - 2b^2\kappa}}, w = \frac{1}{b}\sqrt{a^2 - 2b^2\kappa}, \tag{3.3}$$

$$B_1 = \frac{1}{w}\left(a\xi\kappa + b\kappa'\xi + b\kappa\xi'\right), B_2 = \frac{1}{w}\left((a + b\kappa)\xi' + (\kappa' + \kappa)b\xi\right),$$

$$B_3 = \frac{1}{w}\left(b\xi\tau\right), B_4 = -\frac{1}{w}\left(a\xi + b\xi'\right) \tag{3.4}$$

and

$$\begin{aligned}\Upsilon_1 &= -(B_1 + \frac{a}{2b}\left(2B_1B_4 + B_2^2 + B_3^2\right)),\\ \Upsilon_2 &= -(B_2 + \frac{1}{2}\left(2B_1B_4 + B_2^2 + B_3^2\right)),\\ \Upsilon_3 &= -B_3, \Upsilon_4 = -B_4.\end{aligned} \tag{3.5}$$

ii) *The cone curvatures* $\kappa_{\gamma_{x\alpha}}(s^*)$ *and* $\tau_{\gamma_{x\alpha}}(s^*)$ *of the curve* $\gamma_{x\alpha}$ *is given by*

$$\kappa_{\gamma_{x\alpha}}(s^*) = -\frac{1}{2}\left(2B_1B_4 + B_2^2 + B_3^2\right)$$

$$\tau_{\gamma_{x\alpha}}(s^*) = \sqrt{2(\Upsilon_1 - \kappa')\Upsilon_4 + (\Upsilon_2 - \kappa)^2 + \Upsilon_3^2 - \kappa_{\gamma_{x\alpha}}^2}, \tag{3.6}$$

where

$$s^* = \frac{1}{b}\int \sqrt{a^2 - 2b^2\kappa(s)}ds.$$

Proof. **i**) We assume that the curve x is a unit speed spacelike curve with the asymptotic orthonormal frame $\{x, \alpha, \beta, y\}$ and cone curvature κ, τ. Differentiating the equation (3.1) with respect to s and considering (2.1), we have

$$\gamma'_{x\alpha}(s^*) = (a\xi)\,\alpha(s) + (b\kappa\xi)x(s) + (-b\xi)\,y(s), \tag{3.7}$$

where

$$\frac{ds^*}{ds} = \frac{1}{b}\sqrt{a^2 - 2b^2\kappa(s)}, \tag{3.8}$$

$$\xi = \frac{1}{\sqrt{a^2 - 2\kappa(s)b^2}}. \tag{3.9}$$

It can be easily seen that the tangent vector $\gamma'_{x\alpha}(s^*) = \alpha_{x\alpha}(s^*)$ is a unit spacelike vector. Differentiating (3.7), we obtain equation as follows

$$\gamma''_{x\alpha}(s^*) = B_1x(s) + B_2\alpha(s) + B_3\beta(s) + B_4y(s), \tag{3.10}$$

where

$$\begin{aligned}B_1 &= \frac{1}{w}\left(a\xi\kappa + b\kappa'\xi + b\kappa\xi'\right), B_2 = \frac{1}{w}\left((a + b\kappa)\xi' + (\kappa' + \kappa)b\xi\right),\\ B_3 &= \frac{1}{w}\left(b\xi\tau\right), B_4 = -\frac{1}{w}\left(a\xi + b\xi'\right).\end{aligned}$$

$$y_{x\alpha}(s^*) = -\gamma''_{x\alpha} - \frac{1}{2}\left\langle\gamma''_{x\alpha}, \gamma''_{x\alpha}\right\rangle\gamma_{x\alpha}. \tag{3.11}$$

By the help of previous equation (3.11), we obtain

$$y_{x\alpha}(s^*) = \Upsilon_1x(s) + \Upsilon_2\alpha(s) + \Upsilon_3\beta(s) + \Upsilon_4y(s), \tag{3.12}$$

where $\Upsilon_1 = -(B_1 + \frac{a}{2b}\left(2B_1B_4 + B_2^2 + B_3^2\right)), \Upsilon_2 = -(B_2 + \frac{1}{2}\left(2B_1B_4 + B_2^2 + B_3^2\right)), \Upsilon_3 = -B_3, \Upsilon_4 = -B_4$.

ii) Using equations $\kappa_{\gamma_{x\alpha}}(s^*) = -\frac{1}{2}\left\langle\gamma''_{x\alpha}, \gamma''_{x\alpha}\right\rangle$ and $\tau^2_{\gamma_{x\alpha}}(s^*) = \langle x''' - \kappa\alpha - \kappa'x, x''' - \kappa\alpha - \kappa'x\rangle - \kappa^2_{\gamma_{x\alpha}}(s^*)$. The curvatures $\kappa_{\gamma_{x\alpha}}(s^*)$ and $\tau_{\gamma_{x\alpha}}(s^*)$ of the $\gamma_{x\alpha}(s^*)$ are explicity obtained by

$$\kappa_{\gamma_{x\alpha}}(s^*) = -\frac{1}{2}\left(2B_1B_4 + B_2^2 + B_3^2\right)$$

$$\tau^2_{\gamma_{x\alpha}}(s^*) = 2(\Upsilon_1 - \kappa')\Upsilon_4 + (\Upsilon_2 - \kappa)^2 + \Upsilon_3^2 - \kappa^2_{\gamma_{x\alpha}}. \tag{3.13}$$

Thus, the theorem is proved. □

Definition 3.3. Let x be unit speed spacelike curve lying on $\mathbf{Q}^3$ with the moving asymptotic orthonormal frame $\{x,\alpha,\beta,y\}$. Then, xy-smarandache curve of x is defined by

$$\gamma_{xy}(s^*)=\frac{1}{\sqrt{2ab}}\left(ax(s)+by(s)\right), \tag{3.14}$$

where $a,b\in\mathbb{R}_0^+$.

Theorem 3.4. *Let x be unit speed spacelike curve in $\mathbf{Q}^3$ with the moving asymptotic orthonormal frame $\{x,\alpha,\beta,y\}$ and cone curvature κ and let γ_{xy} be xy-smarandache curve with asymptotic orthonormal frame $\{\gamma_{xy},\alpha_{xy},\beta_{xy},y_{xy}\}$. Then the following relations hold:*

i) *The asymptotic orthonormal frame $\{\gamma_{xy},\alpha_{xy},\beta_{xy},y_{xy}\}$ of the xy-smarandache curve γ_{xy} is given as*

$$\begin{bmatrix}\gamma_{xy}\\ \alpha_{xy}\\ \beta_{xy}\\ y_{xy}\end{bmatrix}=\begin{bmatrix}\frac{a}{\sqrt{2ab}} & 0 & 0 & \frac{b}{\sqrt{2ab}}\\ 0 & \eta_1 & \eta_2 & 0\\ \frac{(\eta_1\kappa+\eta_2\tau)}{w} & \frac{\eta_1'}{w} & \frac{\eta_2'}{w} & \frac{-\eta_1}{w}\\ (\frac{-\eta_1\kappa-\eta_2\tau}{w}-\frac{aC}{2\sqrt{2ab}}) & \frac{-\eta_1'}{w} & \frac{-\eta_2'}{w} & \frac{\eta_1}{w}-\frac{aC}{2\sqrt{2ab}}\end{bmatrix}\begin{bmatrix}x\\ \alpha\\ \beta\\ y\end{bmatrix}, \tag{3.15}$$

where

$$\begin{aligned}\eta_1 &= \frac{a-b\kappa}{w\sqrt{2ab}},\eta_2=\frac{-b\tau}{w\sqrt{2ab}},\\ w &= \frac{ds^*}{ds}=\sqrt{\frac{a}{2b}-\kappa+\frac{b}{2a}(\kappa^2+\tau^2)},\\ C &= \frac{1}{w^2}\left(-2\eta_1(\eta_1\kappa+\eta_2\tau)+(\eta_1')^2+(\eta_2')^2\right).\end{aligned}$$

ii) *The cone curvature $\kappa_{\gamma_{xy}}(s^*)$ and $\tau_{\gamma_{x\alpha}}(s^*)$ of the curve γ_{xy} is given by*

$$\kappa_{\gamma_{xy}}(s^*)=\frac{-C}{2},$$

$$\tau^2_{\gamma_{x\alpha}}(s^*)=2(\eta_1\kappa+\eta_2\tau+\frac{aC}{2\sqrt{2ab}}-\kappa')(\frac{bC}{2\sqrt{2ab}}-\eta_1)+(\eta_1'+\kappa^2)+(\eta_2')^2-\frac{C^2}{4},, \tag{3.16}$$

where

$$s^*=\int\sqrt{\frac{a}{2b}-\kappa+\frac{b}{2a}(\kappa^2+\tau^2)}ds. \tag{3.17}$$

Proof. **i**) We assume that the curve x is a unit speed spacelike curve with the asymptotic orthonormal frame $\{x,\alpha,\beta,y\}$ and cone curvature κ,τ. Differentiating the equation $(\mathbf{3.14})$ with respect to s and considering $(\mathbf{2.1})$, we have

$$\gamma_{xy}'(s^*)\frac{ds^*}{ds}=\frac{(a-b\kappa(s))}{\sqrt{2ab}}\overrightarrow{\alpha}(s)-\frac{b\tau}{\sqrt{2ab}}\overrightarrow{\beta}(s) \tag{3.18}$$

or

$$\gamma_{xy}'(s^*)=\eta_1\overrightarrow{\alpha}+\eta_2\overrightarrow{\beta}.$$

By considering $(\mathbf{3.17})$, we get

$$\gamma_{xy}'(s^*)=\alpha(s)=\alpha_{xy}. \tag{3.19}$$

Here, it can be easily seen that the tangent vector $\overrightarrow{\alpha}_{xy}$ is a unit spacelike vector.
Differentiating $(\mathbf{3.19})$ and using $(\mathbf{3.17})$, we obtain

$$\gamma_{xy}''(s^*)=\left(\frac{(\eta_1\kappa+\eta_2\tau)}{w}\right)x(s)+\frac{\eta_1'}{w}\alpha+\frac{\eta_2'}{w}\beta-\frac{\eta_1}{w}y(s). \tag{3.20}$$

By the help of equation $y_{xy}(s^*) = -\gamma''_{xy} - \frac{1}{2}\left\langle \gamma''_{xy}, \gamma''_{xy} \right\rangle \gamma_{xy}$, we write

$$y_{xy}(s^*) = (\frac{-\eta_1\kappa - \eta_2\tau}{w} - \frac{aC}{2\sqrt{2ab}})x(s) - \frac{\eta_1'}{w}\alpha - \frac{\eta_2'}{w}\beta + (\frac{\eta_1}{w} - \frac{aC}{2\sqrt{2ab}})y(s). \quad (3.21)$$

ii)

$$\kappa_{\gamma_{xy}}(s^*) = -\frac{1}{2}\left\langle \gamma''_{xy}, \gamma''_{xy} \right\rangle,$$

$$\tau^2_{\gamma_{xy}}(s^*) = \langle \beta - \kappa\alpha - \kappa' x, \beta - \kappa\alpha - \kappa' x \rangle - \kappa^2_{\gamma_{xy}}. \quad (3.22)$$

By using (3.22), the curvatures $\kappa_{\gamma_{xy}}(s^*)$ and $\tau_{\gamma_{xy}}(s^*)$ of the $\gamma_{xy}(s^*)$ are explicity obtained

$$\begin{aligned} \kappa_{\gamma_{xy}}(s^*) &= -\frac{1}{2}\left\langle \gamma''_{xy}, \gamma''_{xy} \right\rangle = \frac{-C}{2}, \\ \tau^2_{\gamma_{x\alpha}}(s^*) &= 2(\eta_1\kappa + \eta_2\tau + \frac{aC}{2\sqrt{2ab}} - \kappa')(\frac{bC}{2\sqrt{2ab}} - \eta_1) \\ &\quad +(\eta_1' + \kappa^2) + (\eta_2')^2 - \frac{C^2}{4} \end{aligned}$$

□

Definition 3.5. Let x be unit speed spacelike curve lying on $\mathbf{Q}^3$ with the moving asymptotic orthonormal frame $\{x, \alpha, \beta, y\}$. Then, αy-smarandache curve of x is defined by

$$\gamma_{\alpha y}(s^*) = \alpha(s) + \frac{b}{a}y(s), \quad (3.23)$$

where $a, b \in \mathbb{R}_0^+$.

Theorem 3.6. *Let x be unit speed spacelike curve in $\mathbf{Q}^3$ with the moving asymptotic orthonormal frame $\{x, \alpha, \beta, y\}$ and cone curvature κ and let $\gamma_{\alpha y}$ be αy-smarandache curve with asymptotic orthonormal frame $\{\gamma_{\alpha y}, \alpha_{\alpha y}, \beta_{\alpha y}, y_{\alpha y}\}$. Then the following relations hold:*

i) *The asymptotic orthonormal frame $\{\gamma_{\alpha y}, \alpha_{\alpha y}, \beta_{\alpha y}, y_{\alpha y}\}$ of the αy-smarandache curve $\gamma_{\alpha y}$ is given as*

$$\begin{bmatrix} \gamma_{\alpha y} \\ \alpha_{\alpha y} \\ \beta_{\alpha y} \\ y_{\alpha y} \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & \frac{b}{a} \\ \rho_1 & \rho_2 & \rho_3 & \rho_4 \\ \frac{\rho_1' + \kappa\rho_2 + \rho_3\tau}{M} & \frac{\rho_2' + \rho_1 - \kappa\rho_4}{M} & \frac{\rho_3' - \tau\rho_4}{M} & \frac{-\rho_2 + \rho_4}{M} \\ -\frac{\rho_1' + \kappa\rho_2 + \rho_3\tau}{M} & -\frac{\rho_2' + \rho_1 - \kappa\rho_4}{M} - \frac{D}{2} & -\frac{\rho_3' + \tau\rho_4}{M} & -\frac{-\rho_2 + \rho_4}{M} - \frac{bD}{a} \end{bmatrix} \begin{bmatrix} x \\ \alpha \\ \beta \\ y \end{bmatrix}, \quad (3.24)$$

where

$$\rho_1 = \frac{\kappa}{M}, \rho_2 = \frac{-b}{a}\left(\frac{\kappa}{M}\right), \rho_3 = \frac{-b}{a}\left(\frac{\tau}{M}\right), \rho_4 = \frac{1}{M}$$

$$M = \sqrt{\frac{b}{a^2}(\kappa^2 + \tau^2) - 2\kappa} \quad (3.25)$$

$$D = \frac{2}{M^2}(\rho_1' + \kappa\rho_2 + \rho_3\tau)(-\rho_2 + \rho_4) + \frac{1}{M^2}((\rho_2' + \rho_1 - \kappa\rho_4)^2 + (\rho_3' - \tau\rho_4)^2).$$

ii) *The cone curvatures $\kappa_{\gamma_{\alpha y}}(s^*)$ and $\tau_{\gamma_{\alpha y}}(s^*)$ of the curve $\gamma_{\alpha y}$ is given by*

$$\kappa_{\gamma_{\alpha y}}(s^*) = -\frac{D}{2}, \quad (3.26)$$

$$\begin{aligned} \tau^2_{\gamma_{\alpha y}}(s^*) &= 2(-\frac{\rho_1' + \kappa\rho_2 + \rho_3\tau}{M} - \kappa')(-\frac{\rho_2 - \rho_4}{M} + \frac{bD}{a}) \\ &\quad +(\frac{\rho_2' + \rho_1 - \kappa\rho_4}{M} + \frac{D}{2} + \kappa)^2 + (\frac{-\rho_3' + \tau\rho_4}{M})^2 - \frac{D^2}{4}, \end{aligned} \quad (3.27)$$

where

$$s^* = \int \sqrt{\frac{b}{a^2}(\kappa^2 + \tau^2) - 2\kappa ds}. \quad (3.28)$$

Proof. **i**) Let the curve x be a unit speed spacelike curve with the asymptotic orthonormal frame $\{x, \alpha, \beta, y\}$ and cone curvature κ, τ. Differentiating the equation (3.23) with respect to s and considering (2.1), we find

$$\gamma'_{\alpha y}(s^*)\frac{ds^*}{ds} = \kappa\overrightarrow{x(s)} - \frac{b}{c}\kappa\overrightarrow{\alpha(s)} - \frac{b}{a}\tau\overrightarrow{\beta(s)} - \overrightarrow{y(s)}.$$

This can be written as following

$$\alpha_{\alpha y}(s^*)\frac{ds^*}{ds} = \frac{\kappa}{M}\overrightarrow{x(s)} - \frac{b}{c}\frac{\kappa}{M}\overrightarrow{\alpha(s)} - -\frac{b}{a}\frac{\tau}{M}\overrightarrow{\beta(s)} - \frac{1}{M}\overrightarrow{y(s)}, \tag{3.29}$$

where

$$M = \frac{ds^*}{ds} = \sqrt{\frac{b}{a^2}(\kappa^2+\tau^2) - 2\kappa}. \tag{3.30}$$

Differentiating (3.29) and using (3.30), we get

$$\gamma''_{\alpha y} = (\frac{\rho'_1 + \kappa\rho_2 + \rho_3\tau}{M})x + (\frac{\rho'_2 + \rho_1 - \kappa\rho_4}{M})\alpha + (\frac{\rho'_3 - \tau\rho_4}{M})\beta + (\frac{-\rho_2 + \rho_4}{M})y, \tag{3.31}$$

where $\rho_1 = \frac{\kappa}{M}, \rho_2 = \frac{-b}{a}\left(\frac{\kappa}{M}\right), \rho_3 = \frac{-b}{a}\left(\frac{\tau}{M}\right), \rho_4 = \frac{1}{M}$.

$$y_{\alpha y}(s^*) = -\gamma''_{\alpha y} - \frac{1}{2}\left\langle\gamma''_{\alpha y}, \gamma''_{\alpha y}\right\rangle\gamma_{\alpha y} \text{ and } \left\langle\gamma''_{\alpha y}, \gamma''_{\alpha y}\right\rangle = D. \tag{3.32}$$

By the help of equation (3.32), we obtain

$$y_{\alpha y}(s^*) = (-\frac{\rho'_1 + \kappa\rho_2 + \rho_3\tau}{M})x(s) + (-\frac{\rho'_2 + \rho_1 - \kappa\rho_4}{M} - \frac{D}{2})\alpha(s)$$

$$+(-\frac{\rho'_3 + \tau\rho_4}{M})\beta(s) + (-\frac{-\rho_2 + \rho_4}{M} - \frac{bD}{a})y(s), \tag{3.33}$$

ii) Using (3.22), we have (3.26) and (3.27). □

Definition 3.7. Let x be unit speed spacelike curve lying on $\mathbf{Q}^3$ with the moving asymptotic orthonormal frame $\{x, \alpha, \beta, y\}$. Then, $x\beta$-smarandache curve of x is defined by

$$\gamma_{x\beta}(s^*) = \frac{a}{b}x(s) + \beta(s), \tag{3.34}$$

where $a, b \in \mathbb{R}_0^+$.

Theorem 3.8. *Let x be unit speed spacelike curve in $\mathbf{Q}^3$ with the moving asymptotic orthonormal frame $\{x, \alpha, \beta, y\}$ and cone curvature κ and let $\gamma_{x\beta}$ be $x\beta$-smarandache curve with asymptotic orthonormal frame $\{\gamma_{x\beta}, \alpha_{x\beta}, \beta_{x\beta}, y_{x\beta}\}$. Then the following relations hold:*

i) *The asymptotic orthonormal frame $\{\gamma_{x\beta}, \alpha_{x\beta}, \beta_{x\beta}, y_{x\beta}\}$ of the $x\beta$-smarandache curve $\gamma_{x\beta}$ is given as*

$$\begin{bmatrix}\gamma_{x\beta}\\ \alpha_{x\beta}\\ \beta_{x\beta}\\ y_{x\beta}\end{bmatrix} = \begin{bmatrix}\frac{a}{b} & 0 & 1 & 0\\ \frac{b}{a}\tau & 1 & 0 & 0\\ \frac{b}{a}\kappa + (\frac{b}{a})^2\tau' & \tau\left(\frac{b}{a}\right)^2 & 0 & -\frac{b}{a}\\ -\left(\frac{b}{a}\right)^3\tau^2 & -\left(\frac{b}{a}\right)^3\tau & M & \frac{b}{a}\end{bmatrix}\begin{bmatrix}x\\ \alpha\\ \beta\\ y\end{bmatrix}, \tag{3.35}$$

where

$$M = -\frac{b^4}{2a^4}\tau^2 + \frac{b^2}{a^2}\kappa + \frac{b^3}{a^3}\tau'. \tag{3.36}$$

ii) *The cone curvatures $\kappa_{\gamma_{x\beta}}(s^*)$ and $\tau_{\gamma_{x\beta}}(s^*)$ of the curve $\gamma_{x\beta}$ is given by*

$$\kappa_{\gamma_{x\beta}}(s^*) = -\frac{b^2}{a^2}\left(\frac{b^2}{a^2}\tau^2 - 2\kappa - 2\frac{b}{a}\tau'\right) \tag{3.37}$$

$$\tau^2_{\gamma_{x\beta}}(s^*) = M^2 - 2\frac{b}{a}\kappa' + \kappa^2 - 6\frac{b^2}{a^2}\kappa + 4\frac{b^4}{a^4} - 2\frac{b^3}{a^3}\tau', \tag{3.38}$$

where

$$s^* = \frac{a}{b}s + A;\ a, b, A \in \mathbb{R}_0^+. \tag{3.39}$$

Proof. **i**) Differentiating the equation (3.34) with respect to s and considering (2.1), we find

$$\gamma'_{x\beta}(s^*)\frac{ds^*}{ds} = \frac{a}{b}\alpha(s) + \tau x(s). \tag{3.40}$$

This can be written as follows

$$\alpha_{x\beta}(s^*) = \frac{b\tau}{a}\overrightarrow{x(s)} + \overrightarrow{\alpha(s)}, \tag{3.41}$$

where

$$\frac{ds^*}{ds} = \frac{a}{b}. \tag{3.42}$$

Differentiating (3.41) and using (3.42), we get

$$\gamma''_{x\beta}(s^*) = (\frac{b}{a}\kappa + (\frac{b}{a})^2\tau')x(s) + (\tau\left(\frac{b}{a}\right)^2)\alpha(s) - \frac{b}{a}y(s)$$

$$y_{x\beta}(s^*) = -\gamma''_{x\beta} - \frac{1}{2}\left\langle\gamma''_{x\beta}, \gamma''_{x\beta}\right\rangle\gamma_{x\beta}. \tag{3.43}$$

By the help of equation (3.43), we obtain

$$y_{x\beta}(s^*) = (-\left(\frac{b}{a}\right)^3\tau^2)x(s) + (-\left(\frac{b}{a}\right)^3\tau)\alpha(s) + M\beta(s) + (\frac{b}{a})y(s), \tag{3.44}$$

where $M = -\frac{b^4}{2a^4}\tau^2 + \frac{b^2}{a^2}\kappa + \frac{b^3}{a^3}\tau'$.

ii) Using (3.22), we have (3.36) and (3.37). □

Definition 3.9. Let x be unit speed spacelike curve lying on $\mathbf{Q}^3$ with the moving asymptotic orthonormal frame $\{x, \alpha, \beta, y\}$. Then, $\alpha\beta$-smarandache curve of x is defined by

$$\gamma_{\alpha\beta}(s^*) = \frac{1}{\sqrt{a^2+b^2}}\left(a\alpha(s) + b\beta(s)\right), \tag{3.45}$$

where $a, b \in \mathbb{R}_0^+$.

Theorem 3.10. *Let x be unit speed spacelike curve in $\mathbf{Q}^3$ with the moving asymptotic orthonormal frame $\{x, \alpha, \beta, y\}$ and cone curvature κ and τ let $\gamma_{\alpha\beta}$ be $\alpha\beta$-smarandache curve with asymptotic orthonormal frame $\{\gamma_{\alpha\beta}, \alpha_{\alpha\beta}, \beta_{\alpha\beta}, y_{\alpha\beta}\}$. Then the following relations hold:*

i) *The asymptotic orthonormal frame $\{\gamma_{\alpha\beta}, \alpha_{\alpha\beta}, \beta_{\alpha\beta}, y_{\alpha\beta}\}$ of the $\alpha\beta$-smarandache curve $\gamma_{\alpha\beta}$ is given as*

$$\begin{bmatrix}\gamma_{\alpha\beta}\\ \alpha_{\alpha\beta}\\ \beta_{\alpha\beta}\\ y_{\alpha\beta}\end{bmatrix} = \begin{bmatrix} 0 & \frac{a}{\sqrt{a^2+b^2}} & \frac{b}{\sqrt{a^2+b^2}} & 0\\ Y_1 & 0 & 0 & Y_2\\ \frac{Y_1'}{E} & \frac{Y_1-\kappa Y_2}{E} & \frac{-\tau Y_2}{E} & \frac{Y_2'}{E}\\ \frac{-Y_1'}{E} & -\frac{Y_1-\kappa Y_2}{E} - \frac{a}{2\sqrt{a^2+b^2}}L & \frac{\tau Y_2}{E} - \frac{b}{2\sqrt{a^2+b^2}}L & \frac{-Y_2'}{E}\end{bmatrix}\begin{bmatrix}x\\ \alpha\\ \beta\\ y\end{bmatrix}, \tag{3.46}$$

where

$$E = \frac{ds^*}{ds} = \sqrt{\frac{2}{a^2+b^2}\mid -a\left(a\kappa + b\tau\right)\mid}, \tag{3.47}$$

$$Y_1 = \frac{a\kappa + b\tau}{E\sqrt{a^2+b^2}}, Y_2 = \frac{-a}{E\sqrt{a^2+b^2}}, \tag{3.48}$$

$$L = \frac{1}{E^2}\left(2Y_1'Y_2' + (Y_1 - \kappa Y_2)^2 + \tau^2 Y_2^2\right). \tag{3.49}$$

ii) *The cone curvatures $\kappa_{\gamma_{\alpha\beta}}(s^*)$ and $\tau_{\gamma_{\alpha\beta}}(s^*)$ of the curve $\gamma_{\alpha\beta}$ is given by*

$$\kappa_{\gamma_{\alpha\beta}}(s^*) = -\frac{L}{2}, \tag{3.50}$$

$$\tau^2_{\gamma_{\alpha\beta}}(s^*) = 2(\frac{Y_1'}{E} + \kappa')(-\frac{Y_2'}{E}) + (\frac{\kappa Y_2 - Y_1}{E} - \frac{a}{2\sqrt{a^2+b^2}}L - \kappa)^2$$

$$+(\frac{\tau Y_2}{E} - \frac{b}{2\sqrt{a^2+b^2}}L)^2 - \frac{L^2}{4}, \tag{3.51}$$

where

$$s^* = \int \sqrt{\frac{2}{a^2+b^2} \mid -a\,(a\kappa + b\tau) \mid} ds. \tag{3.52}$$

Proof. **i**) Let the curve x be a unit speed spacelike curve with the asymptotic orthonormal frame $\{x, \alpha, \beta, y\}$ and cone curvature κ, τ. Differentiating the equation (3.45) with respect to s and considering (2.1), we find

$$\gamma'_{\alpha\beta}(s^*)\frac{ds^*}{ds} = \frac{a\kappa + b\tau}{\sqrt{a^2+b^2}}\overrightarrow{x(s)} - \frac{a}{\sqrt{a^2+b^2}}\overrightarrow{y(s)}, \tag{3.53}$$

where

$$E = \frac{ds^*}{ds} = \sqrt{\frac{2}{a^2+b^2} \mid -a\,(a\kappa + b\tau) \mid}.$$

Differentiating (3.53) and using (3.47), we get

$$\gamma''_{\alpha\beta}(s^*) = (\frac{Y_1'}{E})x(s) + (\frac{Y_1 - \kappa Y_2}{E})\alpha(s) + (\frac{-\tau Y_2}{E})\beta(s) + (\frac{Y_2'}{E})y(s), \tag{3.54}$$

where $Y_1 = \frac{a\kappa+b\tau}{E\sqrt{a^2+b^2}}, Y_2 = \frac{-a}{E\sqrt{a^2+b^2}}$.

$$y_{\alpha\beta}(s^*) = -\gamma''_{\alpha\beta} - \frac{1}{2}\left\langle \gamma''_{\alpha\beta}, \gamma''_{\alpha\beta} \right\rangle \gamma_{\alpha\beta}, \text{ and } \left\langle \gamma''_{\alpha\beta}, \gamma''_{\alpha\beta} \right\rangle = L. \tag{3.55}$$

By the help of equation (3.55), we obtain

$$y_{\alpha\beta}(s^*) = (\frac{-Y_1'}{E})x + (\frac{\kappa Y_2 - Y_1}{E} - \frac{aL}{2\sqrt{a^2+b^2}})\alpha,$$

$$+(\frac{\tau Y_2}{E} - \frac{bL}{2\sqrt{a^2+b^2}})\beta + (\frac{-Y_2'}{E})y, \tag{3.56}$$

ii) Using (3.22), we have (3.50) and (3.51). □

Definition 3.11. Let x be unit speed spacelike curve lying on $\mathbf{Q}^3$ with the moving asymptotic orthonormal frame $\{x, \alpha, \beta, y\}$. Then, βy-smarandache curve of x is defined by

$$\gamma_{\beta y}(s^*) = \beta(s) + \frac{b}{a}y(s), \tag{3.57}$$

where $a, b \in \mathbb{R}_0^+$.

Theorem 3.12. *Let x be unit speed spacelike curve in $\mathbf{Q}^3$ with the moving asymptotic orthonormal frame $\{x, \alpha, \beta, y\}$ and cone curvature κ and let $\gamma_{\beta y}$ be βy-smarandache curve with asymptotic orthonormal frame $\{\gamma_{\beta y}, \alpha_{\beta y}, \beta_{\beta y}, y_{\beta y}\}$. Then the following relations hold:*

i) *The asymptotic orthonormal frame $\{\gamma_{\beta y}, \alpha_{\beta y}, \beta_{\beta y}, y_{\beta y}\}$ of the βy-smarandache curve $\gamma_{\beta y}$ is given as*

$$\begin{bmatrix} \gamma_{\beta y} \\ \alpha_{\beta y} \\ \beta_{\beta y} \\ y_{\beta y} \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 & \frac{b}{a} \\ \frac{a\tau}{b\sqrt{\kappa^2+\tau^2}} & -\frac{\kappa}{\sqrt{\kappa^2+\tau^2}} & -\frac{\tau}{\sqrt{\kappa^2+\tau^2}} & 0 \\ \frac{\omega_1'+\kappa\omega_2+\omega_3\tau}{Z} & \frac{\omega_1+\omega_2'}{Z} & \frac{\omega_3'}{Z} & -\frac{\omega_2}{Z} \\ -\frac{\omega_1'+\kappa\omega_2+\omega_3\tau}{Z} & -\frac{\omega_1+\omega_2'}{Z} & -\frac{\omega_3'}{Z} - \frac{F}{2} & \frac{\omega_2}{Z} - \frac{bF}{2a} \end{bmatrix} \begin{bmatrix} x \\ \alpha \\ \beta \\ y \end{bmatrix}, \tag{3.58}$$

where

$$Z = \frac{ds^*}{ds} = \frac{b}{a}\sqrt{\kappa^2 + \tau^2}, \tag{3.59}$$

$$\omega_1 = \frac{a\tau}{b\sqrt{\kappa^2+\tau^2}}, \omega_2 = -\frac{\kappa}{\sqrt{\kappa^2+\tau^2}}, \omega_3 = -\frac{\tau}{\sqrt{\kappa^2+\tau^2}} \tag{3.60}$$

ii) *The cone curvatures $\kappa_{\gamma_{\beta y}}(s^*)$ and $\tau_{\gamma_{\beta y}}(s^*)$ of the curve $\gamma_{\beta y}$ is given by*

$$\kappa_{\gamma_{x\beta}}(s^*) = -\frac{F}{2} \tag{3.61}$$

$$\tau^2_{\gamma_{\beta y}}(s^*) = 2\left(\frac{\omega_1' + \kappa\omega_2 + \omega_3\tau}{Z} + \kappa'\right)\left(\frac{bF}{2a} - \frac{\omega_2}{Z}\right) + \left(\frac{\omega_1 + \omega_2'}{Z} + \kappa\right)^2 + \frac{F^2}{4}, \tag{3.62}$$

where

$$s^* = \frac{a}{b}\sqrt{\kappa^2+\tau^2};\ a, b, \in \mathbb{R}_0^+. \tag{3.63}$$

Proof. **i**) Differentiating the equation (3.57) with respect to s and considering (2.1), we find

$$\gamma'_{\beta y}(s^*)\frac{ds^*}{ds} = \tau x(s) - \frac{b}{a}\kappa\alpha(s) - \frac{b}{a}\tau\beta(s). \tag{3.64}$$

This can be written as follows

$$\alpha_{\beta y}(s^*) = \frac{a\tau}{b\sqrt{\kappa^2+\tau^2}}x(s) - \frac{\kappa}{\sqrt{\kappa^2+\tau^2}}\alpha(s) - \frac{\tau}{\sqrt{\kappa^2+\tau^2}}\beta(s), \tag{3.65}$$

where

$$\frac{ds^*}{ds} = \frac{b}{a}\sqrt{\kappa^2+\tau^2}. \tag{3.66}$$

Differentiating (3.65) and using (3.66), we get

$$\gamma''_{\beta y}(s^*) \;=\; (\frac{\omega_1' + \kappa\omega_2 + \omega_3\tau}{Z})x(s) + (\frac{\omega_1 + \omega_2'}{Z})\alpha(s) + \left(\frac{\omega_3'}{Z}\right)\beta(s) + \left(-\frac{\omega_2}{Z}\right)y(s),$$

$$y_{\beta y}(s^*) = -\gamma''_{\beta y} - \frac{1}{2}\left\langle \gamma''_{\beta y}, \gamma''_{\beta y}\right\rangle \gamma_{\beta y}. \tag{3.67}$$

By the help of equation (3.67), we obtain

$$y_{\beta y}(s^*) = (-\frac{\omega_1' + \kappa\omega_2 + \omega_3\tau}{Z})x(s) + (-\frac{\omega_1 + \omega_2'}{Z})\alpha(s)$$

$$+\left(-\frac{\omega_3'}{Z} - \frac{F}{2}\right)\beta(s) + (\frac{\omega_2}{Z} - \frac{bF}{2a})y(s), \tag{3.68}$$

where $Z = \frac{ds^*}{ds} = \frac{b}{a}\sqrt{\kappa^2+\tau^2}$.

ii) Using (3.22), we have (3.61) and (3.62). □

Example 3.13. Let x be a spacelike curve in $\mathbf{Q}^3$ with arc length parameter s given by

$$x(s) = (\sin s, \cos s, 0, 1).$$

Then we can write the smarandache curves of the x-curve as follows:

1) $x\alpha$- smarandache curve $\gamma_{x\alpha}$ is given by $\gamma_{x\alpha}(s) = \left(\frac{a}{b}\sin s + \cos s, \frac{a}{b}\cos s - \sin s, 0, \frac{a}{b}\right)$

2) $x\beta$-smarandache curve $\gamma_{x\beta}$ is given by $\gamma_{x\beta}(s) = \left((\frac{a}{b} - 1)\sin s, (\frac{a}{b} - 1)\cos s, 0, \frac{a}{b}\right)$

3) xy- smarandache curve γ_{xy} is given by $\gamma_{xy}(s) = \left(\frac{a}{\sqrt{ab}}\sin s - \cos s, \frac{a}{\sqrt{ab}}\cos s + \sin s, 0, \frac{a}{\sqrt{ab}}\right)$

4) αy- smarandache curve $\gamma_{\alpha y}$ is given by $\gamma_{\alpha y}(s) = \left((1 - \frac{a}{b})\cos s, (1 - \frac{a}{b})\sin s, 0, 0\right)$

5) $\alpha\beta$-smarandache curve $\gamma_{\alpha\beta}$ is given by $\gamma_{\alpha\beta}(s) = \frac{-1}{\sqrt{a^2+b^2}}(b\sin s - a\cos s, a\sin s + b\cos s, 0, 0)$

6) βy-smarandache curve $\gamma_{\beta y}$ is given by $\gamma_{\beta y}(s) = \left(-\sin s - \frac{a}{b}\cos s, -\cos s + \frac{a}{b}\sin s, 0, 0\right)$,

where $a, b \in \mathbb{R}_0^+$.

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