Applied Mathematics E-Notes, 16(2016), 198-209 © ISSN 1607-2510
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Integral Invariants Of Mannheim Offsets Of Ruled Surfaces*

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Received 26 April 2016

Abstract

In this study, we identify the instantaneous Pfaffian vector of the closed motion defined along the striction curve of the ruled surface φ^* which is a Mannheim offset of closed ruled surface φ in $\mathbb{E}^3$. Using this vector, we get the Steiner rotation vector and the Steiner translation vector of this motion and give new characteristic results about the pitch and the angle of the pitch which are the integral invariants of Mannheim offsets of ruled surface.

1 Introduction

Ruled surface was found and investigated by Gaspard Monge who established the partial differential equation that satisfies all ruled surface. In differential geometry, the ruled surface have been treated in different ways. These surfaces can be generated by moving a line along a chosen curve or formed by one parameter set of lines [1, 2]. The ruled surfaces are one of the easiest of all surfaces to parametrize. The ruled surfaces are very important in many areas of sciences for instance Computer-Aided Geometric Design (CAGD), Computer-Aided Manufacturing (CAM), kinematics and geometric modelling.

From past to today, characteristic properties of the ruled surfaces and their integral invariants have been examined in Euclidean and non-Euclidean spaces, [3–19]. Müller showed that the pitch of closed ruled surface is integral invariant [3]. Ravani and Ku studied Bertrand offsets of ruled surface in [5]. Based on this study, Kasap and Kuruoğlu gave integral invariants of Bertrand offsets of ruled surface in [6]. Moreover, the involute-evolute offsets of ruled surface is defined by Kasap et al. in [7] and Mannheim offsets of ruled surface is defined by Orbay et al. in [8]. These offset surfaces are defined using the geodesic Frenet frame which was given in [4, 5]. According to the involute-evolute offsets of ruled surface [7] Şentürk and Yüce have calculated integral invariants of these offsets with respect to the geodesic Frenet frame [9].

*Mathematics Subject Classifications: 53A05, 53A17, 53A23.

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Using the directions defined on a surface or a curve, a wide range of offsets of that surface or that curve can be found out and Bertrand, Involute-Evolute, Mannheim and Smarandache are cited as samples for this kind of offset. Mannheim curves have been investigated in the Euclidean space $\mathbb{E}^3$ by Liu and Wang in [10, 11] and Mannheim offsets of ruled surface in the Euclidean space $\mathbb{E}^3$ have been considered by Orbay et al. in [8]. It is shown that a theory similar to that of the Mannheim partner curves can be developed for ruled surface.

In this paper, based on Mannheim offsets of ruled surface in [8] we have calculated the instantaneous Pfaffian vector, the Steiner rotation vector, the Steiner translation vector, the pitch and the angle of the pitch of these offsets according to the geodesic Frenet frame.

2 Preliminaries

In this section, we will present some basic concepts related to ruled surface, geodesic Frenet frame and Mannheim offsets of ruled surface.

2.1 Differential Geometry of Ruled Surface with Geodesic Frenet Frame

A ruled surface M in $\mathbb{E}^3$ is generated by a one-parameter family of straight lines. The straight lines are called the *rulings*. The equation of the ruled surface can be written as,

$$\varphi(s,v) = \alpha(s) + ve(s) \text{ and } \|e(s)\| = 1$$

where (α) is a curve which is called the *base curve* of the ruled surface, s is the arc-length parameter of $\alpha(s)$ and the curve which is drawn by $e(s)$ on the unit sphere S^2 is called the *spherical indicatrix curve* and $\mathbf{e}$ is also called the *spherical indicatrix vector* of the ruled surface, [4, 5].

If ruled surface satisfies the condition $\varphi(s+P,v) = \varphi(s,v)$ for all $s \in I$, then ruled surface is called *closed*.

The unit normal vector of φ along a general generator $l = \varphi(s_0, v)$ of the ruled surface approaches a limiting direction as v infinitely decreases. This direction is called the *asymptotic normal direction* and defined as by [4, 5]

$$\mathbf{g}(s) = \frac{\mathbf{e} \times \mathbf{e}_s}{\|\mathbf{e}_s\|} \text{ where } \mathbf{e}_s = \frac{d\mathbf{e}}{ds}.$$

The point on which the unit normal vector of φ is perpendicular to $\mathbf{g}$ asymptotic normal direction is called the *striction point (or central point)* on l and the curve drawn by these points are called the *striction curve* of φ [4, 5]. The striction curve of the ruled surface φ can be written as

$$c(s) = \alpha(s) - \frac{\langle \alpha_s, \mathbf{e}_s \rangle}{\langle \mathbf{e}_s, \mathbf{e}_s \rangle} \mathbf{e}(s).$$

In this case, we will take the striction curve as the base curve of the ruled surface. So the ruled surface can be defined as

$$\varphi(s,v) = c(s) + ve(s).$$

The direction of the unit normal at a striction point is called the *central normal* of the ruled surface φ and it is calculated by [4, 5]

$$\mathbf{t} = \frac{\mathbf{e}_s}{\|\mathbf{e}_s\|}.$$

The orthonormal system $\{\mathbf{e}, \mathbf{t}, \mathbf{g}\}$ is called the *geodesic Frenet frame* of the ruled surface φ such that [4, 5]

$$\begin{cases} \mathbf{e} = \mathbf{e}, \\ \mathbf{t} = \frac{\mathbf{e}_s}{\|\mathbf{e}_s\|}, \\ \mathbf{g} = \frac{\mathbf{e}\times\mathbf{e}_s}{\|\mathbf{e}\times\mathbf{e}_s\|}. \end{cases} \tag{1}$$

The derivative formulae of the geodesic Frenet frame are given as follows [4, 5]

$$\begin{cases} \mathbf{e}_q = \mathbf{t}, \\ \mathbf{t}_q = \gamma\mathbf{g} - \mathbf{e}, \\ \mathbf{g}_q = -\gamma\mathbf{t}, \end{cases} \tag{2}$$

where q is called the *arc-parameter* of spherical indicatrix curve $(\mathbf{e})$ and γ is called the *geodesic curvature* of $(\mathbf{e})$ with respect to the unit sphere S^2.

Similarly, if we differentiate the equation (1) with respect to the arc-parameter of α, we can give the following equations [4, 5]:

$$\begin{cases} \mathbf{e}_s = q_s\mathbf{t}, \\ \mathbf{t}_s = -q_s\mathbf{e} + \gamma q_s\mathbf{g}, \\ \mathbf{g}_s = -\gamma q_s\mathbf{t}. \end{cases}$$

These equations can be considered the analogue of the equation (2). Moreover, the above equation system can be written as a matrix form:

$$\begin{bmatrix} d\mathbf{e} \\ d\mathbf{t} \\ d\mathbf{g} \end{bmatrix} = \begin{bmatrix} 0 & q_s & 0 \\ -q_s & 0 & \gamma q_s \\ 0 & -\gamma q_s & 0 \end{bmatrix} \begin{bmatrix} \mathbf{e} \\ \mathbf{t} \\ \mathbf{g} \end{bmatrix} \tag{3}$$

with the aim of this matrix, the Pfaffian forms (connection forms) of the geodesic Frenet frame $\{\mathbf{e}, \mathbf{t}, \mathbf{g}\}$ can be obtained such that $w_1 = \gamma q_s$, $w_2 = 0$ and $w_3 = q_s$ [6].

COROLLARY 1 ([5]). The *geodesic curvature* of $(\mathbf{e})$ with respect to the unit sphere S^2 is obtained by the equation (2)

$$\gamma = \frac{\langle \mathbf{e}, \mathbf{e}_s \times \mathbf{e}_{ss} \rangle}{\|\mathbf{e}_s\|^3}.$$

If the successive rulings intersect, the ruled surface is called *developable*. The *distribution parameter* of the ruled surface is defined by

$$P_{\mathbf{e}} = \frac{\det(\alpha_s, \mathbf{e}, \mathbf{e}_s)}{\langle \mathbf{e}_s, \mathbf{e}_s \rangle}.$$

THEOREM 1 ([5]). The ruled surface is developable if and only if $P_e = 0$.

THEOREM 2 ([5]). Let (c) be a striction curve of a developable surface with direction $\mathbf{e}$. The spherical indicatrix, $\mathbf{e}$, is tangent of its striction curve.

2.2 Mannheim Offsets of Ruled Surfaces

The ruled surface φ^* is said to be Mannheim offset of the ruled surface φ if there exists a one to one correspondence between their rulings such that the asymptotic normal of φ and the central normal of φ^* are linearly dependent at the striction points of their corresponding rulings. The base ruled surface $\varphi(s, v)$, can be expressed as

$$\varphi(s, v) = c(s) + ve(s) \ \text{and} \ \|e(s)\| = 1,$$

where (c) is its striction curve and s is the arc length along (c).

If $\varphi^*$is a Mannheim offset of φ, then we can write

$$\begin{bmatrix} \mathbf{e}^* \\ \mathbf{t}^* \\ \mathbf{g}^* \end{bmatrix} = \begin{bmatrix} \cos\theta & \sin\theta & 0 \\ 0 & 0 & 1 \\ \sin\theta & -\cos\theta & 0 \end{bmatrix} \begin{bmatrix} \mathbf{e} \\ \mathbf{t} \\ \mathbf{g} \end{bmatrix}, \tag{4}$$

where $\{\mathbf{e}, \mathbf{t}, \mathbf{g}\}$ and $\{\mathbf{e}^*, \mathbf{t}^*, \mathbf{g}^*\}$ are the geodesic Frenet frames at the point $c(s)$ and $c^*(s)$ of the striction curves of φ and φ^*, respectively and θ is the angle between $\mathbf{e}$ and $\mathbf{e}^*$ as shown in Figure 1, [8].

The equation of the offset surface φ^* can be written as, in terms of its base surface φ,

$$\varphi^*(s, v) = c^*(s) + ve^*(s) = c(s) + R(s)\,\mathbf{g}(s) + v[\cos\theta\mathbf{e}(s) + \sin\theta\mathbf{t}(s)], \tag{5}$$

where R is the distance between the corresponding striction points of φ and φ^*, [8].

THEOREM 3 ([8]). Let the ruled surface φ^* be Mannheim offset of the ruled surface φ. Then φ is developable if and only if R is constant.

3 Integral Invariants of Mannheim Offsets of Ruled Surfaces

Let the ruled surface φ^* be a Mannheim offset of a ruled surface φ, $\{\mathbf{e}, \mathbf{t}, \mathbf{g}\}$ and $\{\mathbf{e}^*, \mathbf{t}^*, \mathbf{g}^*\}$ be the geodesic Frenet frames at the striction point of the ruled surfaces φ

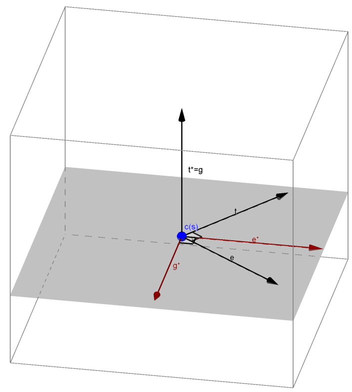

Figure 1: The relation between $\{\mathbf{e},\mathbf{t},\mathbf{g}\}$ frame and $\{\mathbf{e}^*,\mathbf{t}^*,\mathbf{g}^*\}$ frame.

and φ^*. Then, if we differentiate the equation (4) with respect to the arc-parameter of (c), we obtain the following equations

$$\begin{cases} \mathbf{e}^*{}_s = \gamma q_s \sin\theta \mathbf{t}^* - (\theta_s + q_s)\,\mathbf{g}^*, \\ \mathbf{t}^*{}_s = -\gamma q_s \sin\theta \mathbf{e}^* + \gamma q_s \cos\theta \mathbf{g}^*, \\ \mathbf{g}^*{}_s = (\theta_s + q_s)\,\mathbf{e}^* - \gamma q_s \cos\theta \mathbf{t}^*. \end{cases}$$

Thus for the Pfaffian forms (connection forms) of the system $\{\mathbf{e}^*,\mathbf{t}^*,\mathbf{g}^*\}$, we get the following equations

$$\begin{cases} {w_1}^* = \gamma q_s \cos\theta, \\ {w_2}^* = (\theta_s + q_s)\,, \\ {w_3}^* = \gamma q_s \sin\theta. \end{cases} \tag{6}$$

Substituting $w_1 = \gamma q_s$, $w_2 = 0$ and $w_3 = q_s$ the Pfaffian forms (connection forms) of the system $\{\mathbf{e},\mathbf{t},\mathbf{g}\}$ in the equation (6), we can write

$$\begin{cases} {w_1}^* = w_1 \cos\theta, \\ {w_2}^* = \theta_s + w_3, \\ {w_3}^* = w_1 \sin\theta. \end{cases}$$

Let $H^* = sp\,\{\mathbf{e}^*,\mathbf{t}^*,\mathbf{g}^*\}$ be the moving space where $\{\mathbf{e}^*,\mathbf{t}^*,\mathbf{g}^*\}$ is the moving frame along the striction curve (c^*) of φ^* and $H^{*\prime}$ be the fixed space. $H^*/H^{*\prime}$ is used as the motion of H^* according to $H^{*\prime}$. For the instantaneous Pfaffian vector of the motion $H^*/H^{*\prime}$, we get

$$\overrightarrow{w}^* = w_1 \cos\theta \mathbf{e}^* + (\theta_s + w_3)\,\mathbf{t}^* + w_1 \sin\theta \mathbf{g}^*.$$

Let $H = sp\,\{\mathbf{e},\mathbf{t},\mathbf{g}\}$ be the moving space where $\{\mathbf{e},\mathbf{t},\mathbf{g}\}$ is the moving frame along the striction curve (c) of φ , $H^{'}$ be the fixed space and $H/H^{'}$ be a closed motion, then the instantaneous Pfaffian vector of the motion is [6]

$$\overrightarrow{w} = \gamma q_s \mathbf{e} + q_s \mathbf{g} = w_1 \mathbf{e} + w_3 \mathbf{g}.$$

and using this equation we can write w^*

$$\overrightarrow{w}^* = \overrightarrow{w} + \theta_s \mathbf{t}^*.$$

From the Pfaffian forms (connection forms) of the system $\{\mathbf{e}^*, \mathbf{t}^*, \mathbf{g}^*\}$ and the instantaneous Pfaffian vector of the motion $H^*/H^{*'}$, we obtain that

$$\begin{aligned}
\vec{D}^* &= \oint_{(c^*)} \overrightarrow{w^*} \\
&= \oint_{(c+R\mathbf{g})} \vec{w} + \oint_{(c+R\mathbf{g})} \theta_s \mathbf{t}^* \\
&= \vec{D} + \mathbf{e}\left(\oint_{(R\mathbf{g})} w_1\right) + \mathbf{g}\left(\oint_{(R\mathbf{g})} w_3 + \oint_{(c+R\mathbf{g})} \theta_s\right)
\end{aligned}$$

where $\overrightarrow{D} = \oint_{(c)} \overrightarrow{w}$ is the Steiner rotation vector of the motion H/H' which is defined along the striction curve of φ [12].

THEOREM 4. The angle of the pitch of closed Mannheim offsets of ruled surfaces with geodesic Frenet frame which are drawn by the spherical indicatrix vector, the asymptotic normal vector and the central normal vector,respectively, we can write

$$\begin{cases}
\lambda_{e^*} = \cos\theta \lambda_e + \cos\theta \left(\oint_{(R\mathbf{g})} w_1\right), \\
\lambda_{t^*} = \lambda_g + \oint_{(R\mathbf{g})} w_3 + \oint_{(c^*)} \theta_s, \\
\lambda_{g^*} = \sin\theta \lambda_e + \sin\theta \left(\oint_{(R\mathbf{g})} w_1\right).
\end{cases} \tag{7}$$

PROOF. We know that the angle of pitch of closed ruled surface φ with geodesic Frenet frame is calculated by [6]

$$\lambda_e = \oint_{(c)} \left\langle \overrightarrow{D}, \mathbf{e} \right\rangle = \oint_{(c)} w_1. \tag{8}$$

Like that the angle of pitch of closed ruled surface φ with geodesic Frenet frame which is drawn by the central normal [6]

$$\lambda_t = \oint_{(c)} \left\langle \overrightarrow{D}, \mathbf{t} \right\rangle = \oint_{(c)} w_2 = 0 \tag{9}$$

and the angle of pitch of closed ruled surface φ with geodesic Frenet frame which is drawn by the asymptotic normal [6]

$$\lambda_g = \oint_{(c)} \left\langle \overrightarrow{D}, \mathbf{g} \right\rangle = \oint_{(c)} w_3. \tag{10}$$

For the angle of the pitch of closed ruled surface φ^* with geodesic Frenet frame, we can write

$$\begin{aligned}
\lambda_{e^*} &= \left\langle \overrightarrow{D^*}, \mathbf{e}^* \right\rangle = \left\langle \vec{D} + \mathbf{e} \left(\oint_{(R\mathbf{g})} w_1 \right) + \mathbf{g} \left(\oint_{(R\mathbf{g})} w_3 + \oint_{(c+R\mathbf{g})} \theta_s \right), \cos\theta \mathbf{e} + \sin\theta \mathbf{t} \right\rangle \\
&= \cos\theta \lambda_e + \sin\theta \lambda_t + \cos\theta \left(\oint_{(R\mathbf{g})} w_1 \right) \\
&= \cos\theta \lambda_e + \cos\theta \left(\oint_{(R\mathbf{g})} w_1 \right).
\end{aligned}$$

Similarly the angle of the pitch of closed ruled surfaces φ^* with geodesic Frenet frame which are drawn by the asymptotic normal and the central normal,respectively, we can write

$$\begin{aligned}
\lambda_{t^*} &= \left\langle \overrightarrow{D^*}, \mathbf{t}^* \right\rangle = \left\langle \vec{D} + \mathbf{e} \left(\oint_{(R\mathbf{g})} w_1 \right) + \mathbf{g} \left(\oint_{(R\mathbf{g})} w_3 + \oint_{(c+R\mathbf{g})} \theta_s \right), \mathbf{g} \right\rangle \\
&= \left\langle \vec{D}, \mathbf{g} \right\rangle + \oint_{(R\mathbf{g})} w_3 + \oint_{(c^*)} \theta_s \\
&= \lambda_g + \oint_{(R\mathbf{g})} w_3 + \oint_{(c^*)} \theta_s
\end{aligned}$$

and

$$\begin{aligned}\lambda_{g^*} &= \left\langle \overrightarrow{D^*}, \mathbf{g}^* \right\rangle = \left\langle \vec{D} + \mathbf{e}\left(\oint\limits_{(R\mathbf{g})} w_1 \right) + \mathbf{g}\left(\oint\limits_{(R\mathbf{g})} w_3 + \oint\limits_{(c+R\mathbf{g})} \theta_s \right), \sin\theta\mathbf{e} - \cos\theta\mathbf{t} \right\rangle \\ &= \sin\theta \left\langle \vec{D}, \mathbf{e} \right\rangle - \cos\theta \left\langle \vec{D}, \mathbf{t} \right\rangle + \sin\theta \left(\oint\limits_{(R\mathbf{g})} w_1 \right) \\ &= \sin\theta\lambda_e + \sin\theta \left(\oint\limits_{(R\mathbf{g})} w_1 \right).\end{aligned}$$

COROLLARY 2. If we take $\theta = 0$, then we have the following equalities from the equations (7),

$$\begin{cases} \lambda_{e^*} = \lambda_e + \oint\limits_{(R\mathbf{g})} w_1, \\ \lambda_{t^*} = \lambda_g + \oint\limits_{(R\mathbf{g})} w_3, \\ \lambda_{g^*} = 0. \end{cases}$$

COROLLARY 3. If we take $\theta = \pi/2$, then we have the following equalities from the equations (7),

$$\begin{cases} \lambda_{e^*} = 0, \\ \lambda_{t^*} = \lambda_g + \oint\limits_{(R\mathbf{g})} w_3, \\ \lambda_{g^*} = \lambda_e + \oint\limits_{(R\mathbf{g})} w_1. \end{cases}$$

For the Steiner translation vector of the motion $H^*/H^{*'}$, we can write that

$$\overrightarrow{V^*} = \oint\limits_{(c^*)} \overrightarrow{dc^*}$$

and from the equation (5), we have

$$\overrightarrow{V^*} = \oint\limits_{(c+R\mathbf{g})} \overrightarrow{d\,(c + R\mathbf{g})} = \oint\limits_{(c+R\mathbf{g})} \overrightarrow{dc} + R\left(\oint\limits_{(c+R\mathbf{g})} d\mathbf{g} \right) + \mathbf{g}\left(\oint\limits_{(c+R\mathbf{g})} \overrightarrow{dR} \right). \tag{11}$$

and if we substitute the equation (3) in the last equation, we obtain that

$$\overrightarrow{V^*} = \vec{V} - R\lambda_e\mathbf{t} + \oint\limits_{(R\mathbf{g})} \overrightarrow{dc} + \mathbf{g}\left(\oint\limits_{(c+Rg)} \overrightarrow{dR} \right) - \mathrm{R}\mathbf{t}\left(\oint\limits_{(R\mathbf{g})} w_1 \right) \tag{12}$$

where $\overrightarrow{V} = \oint\limits_{(c)} \overrightarrow{dc}$ is the Steiner translation vector [12] of the motion H/H'.

THEOREM 5. The pitch of closed Mannheim offsets of ruled surfaces with geodesic Frenet frame which are drawn by the spherical indicatrix vector, the asymptotic normal vector and the central normal vector, respectively, we can write

$$\begin{cases} L_{e^*} = \cos\theta L_e + \sin\theta L_t + \left\langle \oint\limits_{(R\mathbf{g})} \overrightarrow{dc}, \mathbf{e}^* \right\rangle - R\lambda_{g^*}, \\ L_{t^*} = L_g + \left\langle \oint\limits_{(R\mathbf{g})} \overrightarrow{dc}, \mathbf{t}^* \right\rangle + \oint\limits_{(c^*)} \overrightarrow{dR}, \\ L_{g^*} = \sin\theta L_e - \cos\theta L_t + R\lambda_e^* + \left\langle \oint\limits_{(R\mathbf{g})} \overrightarrow{dc}, \mathbf{g}^* \right\rangle. \end{cases} \tag{13}$$

PROOF. We know that the pitch of closed ruled surface φ with geodesic Frenet frame is calculated by, [6]

$$L_e = \left\langle \overrightarrow{V}, \mathbf{e} \right\rangle = \oint\limits_{(c)} < \mathbf{e}, \overrightarrow{dc} > .$$

Like that the pitch of closed ruled surface φ with geodesic Frenet frame which is drawn by the central normal is calculated by, [6]

$$L_t = \left\langle \overrightarrow{V}, \mathbf{t} \right\rangle = \oint\limits_{(c)} < \mathbf{t}, \overrightarrow{dc} > \tag{14}$$

and the pitch of closed ruled surface φ with geodesic Frenet frame which is drawn by the asymptotic normal is calculated by, [6]

$$L_g = \left\langle \overrightarrow{V}, \mathbf{g} \right\rangle = \oint\limits_{(c)} < \mathbf{g}, \overrightarrow{dc} > . \tag{15}$$

For the pitch of the closed ruled surface φ^* with geodesic Frenet frame, we can write

$$\begin{aligned} L_{e^*} &= \left\langle \overrightarrow{V^*}, \mathbf{e}^* \right\rangle \\ &= \left\langle \vec{V} - R\lambda_e \mathbf{t} + \oint\limits_{(R\mathbf{g})} \overrightarrow{dc} + \mathbf{g}\left(\oint\limits_{(c+R\mathbf{g})} \overrightarrow{dR} \right) - \mathrm{R}\mathbf{t}\left(\oint\limits_{(R\mathbf{g})} w_1 \right), \cos\theta\mathbf{e} + \sin\theta\mathbf{t} \right\rangle \\ &= \cos\theta L_e + \sin\theta L_t + \left\langle \oint\limits_{(R\mathbf{g})} \overrightarrow{dc}, \mathbf{e}^* \right\rangle - R\lambda_{g^*}. \end{aligned}$$

Similarly the pitch of closed ruled surfaces φ^* with geodesic Frenet frame which are drawn by the asymptotic normal and the central normal, respectively, we can write

$$\begin{aligned} L_{t^*} &= \left\langle \overrightarrow{V^*}, \mathbf{t}^* \right\rangle \\ &= \left\langle \vec{V} - R\lambda_e \mathbf{t} + \oint_{(R\mathbf{g})} \overrightarrow{dc} + \mathbf{g}\left(\oint_{(c+R\mathbf{g})} \overrightarrow{dR} \right) - \mathrm{R}\mathbf{t}\left(\oint_{(R\mathbf{g})} w_1 \right), \mathbf{g} \right\rangle \\ &= L_g + \left\langle \oint_{(R\mathbf{g})} \overrightarrow{dc}, \mathbf{t}^* \right\rangle + \oint_{(c^*)} \overrightarrow{dR} \end{aligned}$$

and

$$\begin{aligned} L_{g^*} &= \left\langle \overrightarrow{V^*}, \mathbf{g}^* \right\rangle \\ &= \left\langle \vec{V} - R\lambda_e \mathbf{t} + \oint_{(R\mathbf{g})} \overrightarrow{dc} + \mathbf{g}\left(\oint_{(c+R\mathbf{g})} \overrightarrow{dR} \right) - \mathrm{R}\mathbf{t}\left(\oint_{(R\mathbf{g})} w_1 \right), \sin\theta \mathbf{e} - \cos\theta \mathbf{t} \right\rangle \\ &= \sin\theta L_e - \cos\theta L_t + R\lambda_e^* + \left\langle \oint_{(R\mathbf{g})} \overrightarrow{dc}, \mathbf{g}^* \right\rangle. \end{aligned}$$

COROLLARY 4. If we take $\theta = 0$, then we have the following equalities from the equations (13),

$$\begin{aligned} L_{e^*} &= L_e + \left\langle \oint_{(R\mathbf{g})} \overrightarrow{dc}, \mathbf{e} \right\rangle, \\ L_{t^*} &= L_g + \left\langle \oint_{(R\mathbf{g})} \overrightarrow{dc}, \mathbf{g} \right\rangle + \oint_{(c^*)} \overrightarrow{dR}, \\ L_{g^*} &= -L_t + R(\lambda_e + \oint_{(R\mathbf{g})} w_1) - \left\langle \oint_{(R\mathbf{g})} \overrightarrow{dc}, \mathbf{t} \right\rangle. \end{aligned}$$

COROLLARY 5. If we take $\theta = \pi/2$, then we have the following equalities from the equations (13),

$$\begin{aligned} L_{e^*} &= L_t + \left\langle \oint_{(R\mathbf{g})} \overrightarrow{dc}, \mathbf{t} \right\rangle - R(\lambda_e + \oint_{(R\mathbf{g})} w_1), \\ L_{t^*} &= L_g + \left\langle \oint_{(R\mathbf{g})} \overrightarrow{dc}, \mathbf{g} \right\rangle + \oint_{(c^*)} \overrightarrow{dR}, \\ L_{g^*} &= L_e + \left\langle \oint_{(R\mathbf{g})} \overrightarrow{dc}, \mathbf{e} \right\rangle. \end{aligned}$$

Acknowledgment. The authors would like to thank the referee for a number of valuable suggestions regarding a previous version of this paper. The first author was supported TÜBİTAK - The Scientific and Technological Research Council of Turkey.

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