Scientia Magna
Vol. 3 (2007), No. 2, 25-28

On the Smarandache LCM dual function

Chengliang Tian and Na Yuan

Department of Mathematics, Northwest University
Xi'an, Shaanxi, P.R.China

Received March 13, 2007

Abstract For any positive integer n, the Smarandache LCM dual function $SL^*(n)$ is defined as the greatest positive integer k such that $[1,\ 2,\ \cdots,\ k]$ divides n. The main purpose of this paper is using the elementary method to study the calculating problem of a Dirichlet series involving the Smarandache LCM dual function $SL^*(n)$ and the mean value distribution property of $SL^*(n)$, obtain an exact calculating formula and a sharper asymptotic formula for it.

Keywords Smarandache LCM dual function, Dirichlet series, exact calculating formula, asymptotic formula.

§1. Introduction and result

For any positive integer n, the famous F.Smarandache LCM function $SL(n)$ is defined as the smallest positive integer k such that $n \mid [1,\ 2,\ \cdots,\ k]$, where $[1,\ 2,\ \cdots,\ k]$ denotes the least common multiple of all positive integers from 1 to k. That is,

$$SL(n) = \min\{k:\ k \in N,\ n \mid [1,\ 2,\ \cdots,\ k]\}.$$

About the elementary properties of $SL(n)$, many people had studied it, and obtained some interesting results, see references [1] and [2]. For example, Murthy [1] proved that if n be a prime, then $SL(n) = S(n)$, where $S(n) = \min\{m:\ n|m!,\ m \in N\}$ be the F.Smarandache function. Simultaneously, Murthy [1] also proposed the following problem:

$$SL(n) = S(n), \quad S(n) \neq n\ ? \tag{1}$$

Le Maohua [2] solved this problem completely, and proved the following conclusion:

Every positive integer n satisfying (1) can be expressed as

$$n = 12 \ \texttt{or}\ \ n = p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_r^{\alpha_r} p,$$

where $p_1,\ p_2,\ \cdots,\ p_r,\ p$ are distinct primes and $\alpha_1,\ \alpha_2,\ \cdots,\ \alpha_r$ are positive integers satisfying $p > p_i^{\alpha_i}$, $i = 1,\ 2,\ \cdots,\ r$.

Zhongtian Lv [3] proved that for any real number $x > 1$ and fixed positive integer k, we have the asymptotic formula

$$\sum_{n \leq x} SL(n) = \frac{\pi^2}{12} \cdot \frac{x^2}{\ln x} + \sum_{i=2}^{k} \frac{c_i \cdot x^2}{\ln^i x} + O\left(\frac{x^2}{\ln^{k+1} x}\right),$$

where c_i $(i = 2,\ 3,\ \cdots,\ k)$ are computable constants.

Now, we define the Smarandache LCM dual function $SL^*(n)$ as follows:

$$SL^*(n) = \max\{k:\ k \in N,\ [1,\ 2,\ \cdots,\ k] \mid n\}.$$

For example: $SL^*(1) = 1$, $SL^*(2) = 2$, $SL^*(3) = 1$, $SL^*(4) = 2$, $SL^*(5) = 1$, $SL^*(6) = 3$, $SL^*(7) = 1$, $SL^*(8) = 2$, $SL^*(9) = 1$, $SL^*(10) = 2$, $\cdots$. Obviously, if n is an odd number, then $SL^*(n) = 1$. If n is an even number, then $SL^*(n) \geq 2$. About the other elementary properties of $SL^*(n)$, it seems that none had studied it yet, at least we have not seen such a paper before. In this paper, we use the elementary method to study the calculating problem of the Dirichlet series:

$$\sum_{n=1}^{\infty} \frac{SL^*(n)}{n^s}, \tag{2}$$

and give an exact calculating formula for (2). At the same time, we also study the mean value properties of $SL^*(n)$, and give a sharper mean value formula for it. That is, we shall prove the following two conclusions:

Theorem 1. For any real number $s > 1$, the series (2) is absolutely convergent, and

$$\sum_{n=1}^{\infty} \frac{SL^*(n)}{n^s} = \zeta(s) \sum_{\alpha=1}^{\infty} \sum_{p} \frac{(p^\alpha - 1)(p^s - 1)}{[1,\ 2,\ \cdots,\ p^\alpha]^s},$$

where $\zeta(s)$ is the Riemann zeta-function, $\sum_p$ denotes the summation over all primes.

Theorem 2. For any real number $x > 1$, we have the asymptotic formula

$$\sum_{n \leq x} SL^*(n) = c \cdot x + O(\ln^2 x),$$

where $c = \sum_{\alpha=1}^{\infty} \sum_{p} \frac{(p^\alpha - 1)(p - 1)}{[1,\ 2,\ \cdots,\ p^\alpha]}$ is a constant.

Note that $\zeta(2) = \pi^2/6$, from Theorem 1 we may immediately deduce the identity:

$$\sum_{n=1}^{\infty} \frac{SL^*(n)}{n^2} = \frac{\pi^2}{6} \sum_{\alpha=1}^{\infty} \sum_{p} \frac{(p^\alpha - 1)(p^2 - 1)}{[1,\ 2,\ \cdots,\ p^\alpha]^2}.$$

§2. Some useful lemmas

To complete the proofs of the theorems, we need the following lemmas.

Lemma 1. For any positive integer n, there exist a prime p and a positive integer α such that

$$SL^*(n) = p^\alpha - 1.$$

Proof. Assume that $SL^*(n) = k$. From the definition of the Smarandache LCM dual function $SL^*(n)$ we have

$$[1,\ 2,\ \cdots,\ k] \mid n, \quad (k+1) \dagger n,$$

else $[1,\ 2,\ \cdots,\ k,\ k+1] \mid n$, then $SL^*(n) \geq k+1$. This contradicts with $SL^*(n) = k$. Assume that $k+1 = p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_s^{\alpha_s}$, where p_i is a prime, $p_1 < p_2 < \cdots < p_s$, $\alpha_i \geq 1,\ i = 1,\ 2,\ \cdots,\ s$. If $s > 1$, then $p_1^{\alpha_1} \leq k$, $p_2^{\alpha_2} \cdots p_s^{\alpha_s} \leq k$, so

$$p_1^{\alpha_1} \mid [1, 2, \cdots, k], \quad p_2^{\alpha_2} \cdots p_s^{\alpha_s} \mid [1, 2, \cdots, k].$$

Since $(p_1^{\alpha_1},\ p_2^{\alpha_2} \cdots p_s^{\alpha_s}) = 1$, we have

$$p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_s^{\alpha_s} \mid [1,\ 2,\ \cdots,\ k].$$

Namely, $k+1 \mid [1,\ 2,\ \cdots,\ k]$. From this we deduce that $k+1 \mid n$. This contradicts with $SL^*(n) = k$. Hence $s = 1$. Consequently $k+1 = p^\alpha$. That is, $SL^*(n) = p^\alpha - 1$. This completes the proof of Lemma 1.

Lemma 2. Let $L(n)$ denotes the least common multiple of all positive integers from 1 to n, then we have

$$\ln(L(n)) = n + O\left(n \cdot \exp\left(\frac{-c(\ln n)^{\frac{3}{5}}}{(\ln \ln n)^{\frac{1}{5}}}\right)\right),$$

where c is a positive constant.

Proof. See reference[4].

§3. Proof of the theorems

In this section, we shall complete the proof of the theorems. First we prove Theorem 1.

From the definition of the Smarandache LCM dual function $SL^*(n)$ we know that if $[1,\ 2,\ \cdots,\ k] \mid n$, then $[1,\ 2,\ \cdots,\ k] \leq n$, $\ln([1,\ 2,\ \cdots,\ k]) \leq \ln n$. Hence, from Lemma 2 we have $SL^*(n) = k \leq \ln n$, $\dfrac{SL^*(n)}{n^s} \leq \dfrac{\ln n}{n^s}$. Consequently, if $s > 1$, then the Dirichlet series $\sum\limits_{n=1}^{\infty} \dfrac{SL^*(n)}{n^s}$ is absolutely convergent. From Lemma 1 we know that $SL^*(n) = p^\alpha - 1$, then $[1,\ 2,\ \cdots,\ p^\alpha - 1] \mid n$. Let $n = [1,\ 2,\ \cdots,\ p^\alpha - 1] \cdot m$, then $p \dagger m$, so for any real number $s > 1$, we have

$$\begin{aligned}
\sum_{n=1}^{\infty} \frac{SL^*(n)}{n^s} &= \sum_{\alpha=1}^{\infty} \sum_{p} \sum_{\substack{n=1 \\ SL^*(n)=p^\alpha-1}}^{\infty} \frac{p^\alpha - 1}{n^s} = \sum_{\alpha=1}^{\infty} \sum_{p} \sum_{\substack{m=1 \\ p\dagger m}}^{\infty} \frac{p^\alpha - 1}{[1,\ 2,\ \cdots,\ p^\alpha - 1]^s \cdot m^s} \\
&= \sum_{\alpha=1}^{\infty} \sum_{p} \frac{p^\alpha - 1}{[1,\ 2,\ \cdots,\ p^\alpha - 1]^s} \sum_{\substack{m=1 \\ p\dagger m}}^{\infty} \frac{1}{m^s} \\
&= \sum_{\alpha=1}^{\infty} \sum_{p} \frac{p^\alpha - 1}{[1,\ 2,\ \cdots,\ p^\alpha - 1]^s} \left(\sum_{m=1}^{\infty} \frac{1}{m^s} - \sum_{m=1}^{\infty} \frac{1}{p^s \cdot m^s}\right) \\
&= \sum_{\alpha=1}^{\infty} \sum_{p} \frac{p^\alpha - 1}{[1,\ 2,\ \cdots,\ p^\alpha - 1]^s} \left(\sum_{m=1}^{\infty} \frac{1}{m^s}(1 - \frac{1}{p^s})\right) \\
&= \left(\sum_{m=1}^{\infty} \frac{1}{m^s}\right) \sum_{\alpha=1}^{\infty} \sum_{p} \frac{(p^\alpha - 1)(p^s - 1)}{[1,\ 2,\ \cdots,\ p^\alpha]^s}
\end{aligned}$$

$$= \zeta(s)\cdot\sum_{\alpha=1}^{\infty}\sum_{p}\frac{(p^\alpha-1)(p^s-1)}{[1,\ 2,\ \cdots,\ p^\alpha]^s}.$$

This proves the theorem 1.

From the definition of the Smarandache LCM dual function $SL^*(n)$, Lemma 1 and Lemma 2 we also have

$$\begin{aligned}\sum_{n\le x} SL^*(n) &= \sum_{\substack{[1,\ 2,\ \cdots,\ p^\alpha-1]\cdot m\le x\\ (m,\ p)=1}}(p^\alpha-1) = \sum_{[1,\ 2,\ \cdots,\ p^\alpha-1]\le x}(p^\alpha-1)\sum_{\substack{m\le\frac{x}{[1,\ 2,\ \cdots,\ p^\alpha-1]}\\ p\dagger m}}1\\
&= \sum_{[1,\ 2,\ \cdots,\ p^\alpha-1]\le x}(p^\alpha-1)\left(\frac{x}{[1,\ 2,\ \cdots,\ p^\alpha-1]}-\frac{x}{[1,\ 2,\ \cdots,\ p^\alpha]}+O(1)\right)\\
&= x\cdot\sum_{[1,\ 2,\ \cdots,\ p^\alpha-1]\le x}\frac{(p^\alpha-1)(p-1)}{[1,\ 2,\ \cdots,\ p^\alpha]}+O\left(\sum_{[1,\ 2,\ \cdots,\ p^\alpha-1]\le x}p^\alpha\right)\\
&= x\cdot\sum_{\alpha=1}^{\infty}\sum_{p}\frac{(p^\alpha-1)(p-1)}{[1,\ 2,\ \cdots,\ p^\alpha]}+O\left(\ln^2 x\right)\\
&= c\cdot x+O\left(\ln^2 x\right),\end{aligned}$$

where $c=\sum_{\alpha=1}^{\infty}\sum_{p}\frac{(p^\alpha-1)(p-1)}{[1,\ 2,\ \cdots,\ p^\alpha]}$ is a constant.

This completes the proof of Theorem 2.

References

[1] A. Murthy, Some notions on least common multiples, Smarandache Notions Journal, **12**(2001), 307-309.

[2] Le Maohua, An equation concerning the Smarandache LCM function, Smarandache Notions Journal, **14**(2004), 186-188.

[3] Zhongtian Lv, On the F.Smarandache LCM function and its mean value, Scientia Magna, **3**(2007), No. 1, 22-25.

[4] Pan Xiaowei，On some new Smarandache functions, Mathematics in Practice and Theory，**37**(2007), No.8, 155-158.