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On the F.Smarandache LCM function and its mean value

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Abstract For any positive integer n, the famous F.Smarandache LCM function $SL(n)$ is defined as the smallest positive integer k such that $n \mid [1,\ 2,\ \cdots,\ k]$, where $[1,\ 2,\ \cdots,\ k]$ denotes the least common multiple of 1, 2, $\cdots$, k. The main purpose of this paper is using the elementary methods to study the mean value properties of $\ln SL(n)$, and give a sharper asymptotic formula for it.

Keywords F.Smarandache LCM function, mean value, asymptotic formula.

§1. Introduction and Results

For any positive integer n, the famous F.Smarandache LCM function $SL(n)$ is defined as the smallest positive integer k such that $n \mid [1,\ 2,\ \cdots,\ k]$, where $[1,\ 2,\ \cdots,\ k]$ denotes the least common multiple of 1, 2, $\cdots$, k. For example, the first few values of $SL(n)$ are $SL(1) = 1$, $SL(2) = 2$, $SL(3) = 3$, $SL(4) = 4$, $SL(5) = 5$, $SL(6) = 3$, $SL(7) = 7$, $SL(8) = 8$, $SL(9) = 9$, $SL(10) = 5$, $SL(11) = 11$, $SL(12) = 4$, $SL(13) = 13$, $SL(14) = 7$, $SL(15) = 5$, $\cdots$. About the elementary properties of $SL(n)$, some authors had studied it, and obtained some interesting results, see reference [3] and [4]. For example, Murthy [4] showed that if n is a prime, then $SL(n) = S(n)$, where $S(n)$ denotes the Smarandache function, i.e., $S(n) = \min\{m :\ n|m!,\ m \in N\}$. Simultaneously, Murthy [4] also proposed the following problem:

$$SL(n) = S(n), \quad S(n) \neq n\ ? \tag{1}$$

Le Maohua [5] completely solved this problem, and proved the following conclusion:

Every positive integer n satisfying (1) can be expressed as

$$n = 12 \ \texttt{ or } \ n = p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_r^{\alpha_r} p,$$

where p_1, p_2, $\cdots$, p_r, p are distinct primes, and α_1, α_2, $\cdots$, α_r are positive integers satisfying $p > p_i^{\alpha_i}$, $i = 1, 2, \cdots, r$.

Lv Zhongtian [6] obtained the asymptotic formula:

$$\sum_{n \leq x} SL(n) = \frac{\pi^2}{12} \cdot \frac{x^2}{\ln x} + \sum_{i=2}^{k} \frac{c_i \cdot x^2}{\ln^i x} + O\left(\frac{x^2}{\ln^{k+1} x}\right).$$

In reference [7], Professor Zhang Wenpeng asked us to study the asymptotic properties of $\sum\limits_{n \leq x} \ln SL(n)$. About this problem, it seems that none had studied it, at least we have not seen related papers before. The main purpose of this paper is using the elementary methods to study this problem, and obtain a sharper asymptotic formula for it. That is, we shall prove the following:

Theorem 1. For any real number $x > 1$, we have the asymptotic formula

$$\sum_{n\le x} \ln SL(n) = x\ln x + O\left(x\right).$$

Using the same method of proving Theorem 1 we can also give a similar asymptotic formula for the F.Smarandache function $S(n)$. That is, we have the following:

Theorem 2. For any real number $x > 1$, we have

$$\sum_{n\le x} \ln S(n) = x\ln x + O(x),$$

where $S(n)$ denotes the Smarandache function.

§2. Proof of the theorems

To complete the proof of the theorems, we need the following two important Lemmas.

Lemma 1. For any positive integer $n > 1$, let $n = p_1^{\alpha_1}p_2^{\alpha_2}\cdots p_s^{\alpha_s}$ denotes the factorization of n into prime powers, if $\alpha_1 \ge 2, \alpha_2 \ge 2, \cdots \alpha_s \ge 2$, then we call such an integer n as a square-full number. Let $A_2(x)$ denotes the number of all square-full integers not exceeding x, then we have the asymptotic formula

$$A_2(x) = \frac{\zeta(\frac{3}{2})}{\zeta(3)}x^{\frac{1}{2}} + \frac{\zeta(\frac{2}{3})}{\zeta(2)}x^{\frac{1}{3}} + O\left(x^{\frac{1}{6}}\exp\left(-C\log^{\frac{3}{5}}x(\log\log x)^{-\frac{1}{5}}\right)\right), \tag{2}$$

where $C > 0$ is a constant.

Proof. See reference [8].

Lemma 2. Let p be a prime, k be any positive integer. Then for any real number $x \ge 1$, we have the asymptotic formula

$$\sum_{\substack{pk\le x\\(p,\ k)=1}} \ln p = x\ln x + O\left(x\right).$$

Proof. From the several different forms of the Prime Theorem (See reference [2], [7] and [9]), we know that

$$\sum_{k\le x}\frac{\ln p}{p} = \ln x + O\left(1\right),$$

$$\sum_{k\le x}\ln p = x + O\left(\frac{x}{\ln x}\right),$$

and

$$\sum_{k\le x}\frac{\ln p}{p^2} = D + O\left(\frac{1}{\ln x}\right),$$

where D be an positive constant.

Applying these asymptotic formulas, we have

$$\begin{aligned}\sum_{\substack{pk\leq x\\(p,\ k)=1}}\ln p &= \sum_{p\leq x}\ln p\sum_{\substack{k\leq\frac{x}{p}\\(p,\ k)=1}}1\\ &= \sum_{p\leq x}\ln p\left(\frac{x}{p}-\frac{x}{p^2}+O\left(1\right)\right)\\ &= x\sum_{p\leq x}\frac{\ln p}{p}-x\sum_{p\leq x}\frac{\ln p}{p^2}+O\left(\sum_{p\leq x}\ln p\right)\\ &= x\ln x+O\left(x\right).\end{aligned}$$

This proves Lemma 2.

Now, we use these Lemmas to complete the proof of our theorems. Let $U(n)=\sum\limits_{n\leq x}\ln SL(n)$. First, we estimate the upper bound of $U(n)$. In fact, from the definition of F.Smarandache LCM function $SL(n)$, we know that for any positive integer n, $SL(n)\leq n$ and $\ln SL(n)\leq \ln n$, so we have

$$\sum_{n\leq x}\ln SL(n)\leq\sum_{n\leq x}\ln n.$$

Then from the Euler's summation formula (see reference [2]), we may immediately deduce that

$$U(n)\leq\sum_{n\leq x}\ln n=x\ln x-x+O\left(\ln x\right)=x\ln x+O\left(x\right). \tag{3}$$

Now we estimate the lower bound of $U(n)$. For any positive integer $n>1$, let $n=p_1^{\alpha_1}p_2^{\alpha_2}\cdots p_s^{\alpha_s}$ be the factorization of n into prime power, we divide the interval $[1,\ n]$ into two subsets A and B. A denotes the set of all integers in the interval $[1,\ n]$ such that $\alpha_i\geq 2$ $(i=1,\ 2,\ \cdots,\ s)$. That is to say, A denotes the set of all square-full numbers in the interval $[1,\ n]$; B denotes the set of all integers n with $n\in[1,\ n]$ but $n\notin A$. Then we have

$$U(n)=\sum_{\substack{n\leq x\\n\in A}}\ln SL(n)+\sum_{\substack{n\leq x\\n\in B}}\ln SL(n).$$

From Lemma 1 and the definition of A, we have

$$\sum_{\substack{n\leq x\\n\in A}}\ln SL(n)\leq\sum_{\substack{n\leq x\\n\in A}}\ln n\leq\sum_{\substack{n\leq x\\n\in A}}\ln x=\ln x\sum_{\substack{n\leq x\\n\in A}}1=\ln x\cdot A_2(x)\ll\sqrt{x}\ln x. \tag{4}$$

Now we estimate the summation in set B. Since $SL(n)=\max\{p_1^{\alpha_1},\ p_2^{\alpha_2},\ \cdots,\ p_s^{\alpha_s}\}$ (see reference [4]), so for any $n\in B$, there must exist a prime p such that $p|n$ and $p^2\dagger n$. Therefore, from the definition of $SL(n)$ we have $SL(np)\geq p$. Using this estimate we may immediately deduce that

$$\sum_{\substack{n\leq x\\n\in B}}\ln SL(n)=\sum_{\substack{np\leq x\\(n,\ p)=1}}\ln SL(np)\geq\sum_{\substack{np\leq x\\(n,\ p)=1}}\ln p. \tag{5}$$

Then from Lemma 2 and (5), we have

$$\sum_{\substack{n\le x\\ n\in B}} \ln SL(n) \ge x\ln x + O\left(x\right). \tag{6}$$

Combining (3) and (6), we may immediately deduce the asymptotic formula

$$\sum_{n\le x} \ln SL(n) = x\ln x + O\left(x\right).$$

This completes the proof of Theorem 1.

Using the same method of proving Theorem 1, we can also prove Theorem 2.

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