Ultra Scientist Vol. 26(3)B, 283-289 (2014).

Lagrange's Property For Finite Semigroups

K. JAYSHREE and P. GAJIVARADHAN

(Acceptance Date 10th December, 2014)

Abstract

In this paper non abstract finite semigroups which satisfy classical Lagrange's Theorem is analysed. It is found that finite semigroups in general does not satisfy Lagrange's theorem. So to overcome this problem two new properties satisfied by finite semigroups in terms of subsemigroups are defined. The two new notions are anti Lagrange's property and weak Lagrange's property. Finite semigroups in general satisfy both these properties barring the semigroup S={Z_p, ×, p a prime} which does not satisfy weak Lagrange's property.

Key words: Lagrange's Theorem, nilpotent element, idempotent, anti Lagrange's property, weak Lagrange's property.

1 Introduction

In this paper semigroups are analysed as generalization of groups. Most of the researchers have studied semigroups as the algebraic structure akin to rings. That is why several properties like ideals, idempotents, units, zero divisors enjoyed by rings are studied or analysed for semigroups. Here the study is different and distinct for the study seeks to find out those properties common in semigroups and groups and those properties distinct in semigroups and groups.

At the outset the notion of idempotents and zero divisors of semigroups can by no means by related with groups. Further for group must have identity but there are semigroups which do not have identity. Only monoids have identity. As every group is a semigroup the notion of Smarandache semigroups have been systematically studied by W.B. Vasantha in[3]. Just for the sake of easy reference a Smarandache semigroup S is nothing but a semigroup which has a proper non empty subset A such A under the operations of S is a group. For more about these notions refer[3]. Throughout this paper by a semigroup we mean only semigroups of finite order. This paper has three sections. Section one is introductory in nature. Here in this paper semigroups which satisfy the basic classical theorem for finite groups, *viz*. the Lagrange theorem is analysed. Those semigroups which do not satisfy the Lagrange's theorem for finite groups is analysed in section two. In the section three conclusions derived from the study are given.

2 Finite Semigroups and classical Lagrange's theorem for finite groups – An analysis :

In this section the study of classical theorem for finite groups viz Lagrange's theorem, converse of Lagrange's theorem is analysed in the case of finite semigroups. This study is relevant as semigroups are nothing but a generalization of groups; as every group is a semigroup however a semigroup in general is not a group. Further the study of Smarandache semigroups only finds or characterizes those semigroups which contain subsets which are subgroups under the operations of the semigroup. Now the first classical theorem for finite groups viz. Lagrange's groups is analyzed in case of finite semigroups. First by a few examples then by the resulting theorems.

The Lagrange's theorem for finite group is as follows:

"If G is a finite group and H is a proper subgroup of G then o(H) / O (G). However if t/ o(G); G need not in general have a subgroup of under t – that is the converse is not true."[2].

Example 2.1: Let S = $\{S_{15}, \times\}$ be the semigroup of order 15. x = 14 $\in$S but $14 \times 14 = 1 \pmod{15}$. Thus $B_1 = \{1, 14\}$ is a subsemigroup and o $(B_1) = \{1, 14\}$ is a subsemigroup which is a cyclic group of order 2 and $o(B_1) = 2$ but $2 \nmid 15$.

Take $B_2 = \{1, 10\} \subseteq S$, clearly $10^2 = 100 \pmod{15}$; that is $10^2 = 10 \pmod{15}$, B_2 is a subsemigroup of order two which is not a subgroup of S. Thus $o(B_2) \nmid o(S)$. Thus Lagrange theorem is not true for finite semigroups in general.

Example 2.2: Let S = $\{Z_{10}, \times\}$ be the semigroup. $P_1 = \{0, 1, 5\}$ is a subsemigroup of S such that $o(P_1) \nmid o(S)$. $P_2 = \{0, 6, 5, 1\}$ is again a subsemigroup of S such that $o(P_2) \nmid o(S)$. Thus the classical Lagrange's theorem for finite groups is not true in general for finite groups.

In view of this the following definition is made.

Definition 2.1: Let {S, ×} be a semigroup of finite order say n. If S has at least one subsemigroup B such that B is not a group and B is only a subsemigroup and $o(B) \nmid o(S)$. Then the semigroup S is said to possess anti Lagrange's property.

In the following a class of semigroups which satisfy the anti Lagrange's property is described.

Proposition 2.1: Let S = $\{Z_n, \times\}$ be a semigroup having nilpotent elements of order two and idempotents, then S satisfies anti Lagrange's property.

Proof: Two cases arise, n even or n-odd.

Case i: When n is even. Let $a \in Z_n \setminus \{0\}$ be such that $a^2 = 0$; then the set P = {0, 1, a} is a subsemigroup of order 3 and $3 \nmid n$.

Hence the claim. If $a \in Z_n \setminus \{1,0\}$ is such that $a^2 = a$ then the set $B = \{0, 1, a\} \subseteq \{Z_n, \times\}$ is a subsemigroup and $| B | = 3$ and $3 \nmid n$. Thus for in case of n even the claim is true.

Case ii. Let n be odd. Now let $a \in Z_n \setminus \{0\}$ be such that $a^2 = 0$, then $T = \{0, a\} \in S$ is a subsemigroup of order two and $o(T) \nmid n$. Hence the claim. Let $a_1 \in S$ be an idempotent of S; then $T_1 = \{0, a_1\}$ or $T_2 = \{1, a_1\}$ are subsemigroups of S and $o(T_1) = 2$ (or $o(T_2) = 2$) so $o(T_i) \nmid o(S)$); $i = 1, 2$. Hence the claim.

Thus $S = \{Z_n, \times\}$ satisfies the anti Lagrange property.

This will be illustrated by an example or two.

Example 2.3: Let $S = \{Z_{20}, \times\}$ be the semigroup under $\times$ modulo 20.

Now $5 \in S$ is such that $5^2 \equiv 5$ (mod 20) and $10 \in Z_{20}$ is such that $10^2 = 0$ (mod 20) hence Z_{20} has a nontrivial idempotent and a nilpotent element of order two and $o(Z_{20}) = 20$ that is $n = 20$ is even.

Now $P_1 = \{0, 10, 1\} \subseteq S$ is a subsemigroup of S and $o(P_1) = 3$. Further $3 \nmid 20$. Take $P_2 = \{0, 5, 1\} \subseteq S$; P_2 is a subsemigroup of S and $o(P_2) = 3$ and $3 \nmid 20$, Hence the proposition is verified.

Now other than these take $T_1 = \{0, 5, 10\} \subseteq S$ is again a subsemigroup of S such that $o(T_1) \nmid 20$.

Example 2.4: Let $S = \{Z_{35}, \times\}$ be the semigroup of order 35. $15 \in S$ is such that $15^2 = 15$ (mod 35). Thus $M = \{0, 15, 1\} \subseteq S$ is a subsemigroup of order three. $|M| \nmid o(S)$. $N = \{0, 15\} \subseteq S$ is also a subsemigroup of S and $|N| \nmid o(S)$. $T = \{15, 1\} \subseteq S$ is a subsemigroup of S such that $|T| \nmid o(S)$. Now $7, 5 \in S$ is such that $7 \times 5 = 0$(mod 35) but is not a nilpotent element of order two.

Take $x = 21 \in Z_{35}$, $21^2 \equiv 21$ (mod 35)

$B = \{1, 21, 0\} \subseteq S$ is a subsemigroup of Z_{35} $o(B) \nmid 35$; S satisfies the anti Lagrange property.

Some of these semigroups may not contain nilpotent elements of order two.

Example 2.5: Let $S = \{Z_{25}, \times\}$ be the semigroup of order 25. $x = 5 \in S$ is such that $x^2 = 0$. $M = \{0, 5\} \subseteq S$ is a subsemigroup of order two and $o(M) \nmid o(S)$. Take $N_1 = \{1, 5, 0\} \subseteq S$; N_1 is also a subsemigroup such that $o(N_1) \nmid o(S)$. Now $P = \{0, 10\} \subseteq S$ is a subsemigroup such that $o(P) \nmid o(S)$.

Thus $D = \{0, 5, 10\} \subseteq S$ is a subsemigroup of S such that $o(D) \nmid o(S)$. $E = \{0, 5, 10,1\} \subseteq S$ is a subsemigroup such that $o(E) \nmid o(S)$. Hence S is a semigroup which satisfies the anti Lagrange's property.

In fact there are semigroups which have only idempotents but has no nilpotents elements of order two still those semigroup satisfy anti Lagrange property.

Now for a semigroup S to satisfy weak Lagrange property is made as a definition in the following:

Definition 2.2: Let S be a finite semigroup. If S contains atleast a subsemigroup P which is not a group such that o(P) / o(S) then S is said to satisfy the weak Lagrange's property.

This is illustrated by some examples.

Example 2.6: Let S = $\{Z_{24}, \times\}$ be the semigroup. Let B_1 = {0, 2, 4, 6, 8, 10, 12, 14, 16, 18, 20, 22} $\subseteq$ S; be the subsemigroup. Clearly o(B_1) / o(S). Consider L = {0, 3, 6, 9, 12, 15, 18, 21} $\subseteq$ S be the subsemigroup of S; o(L) / o(S). Now take M = {0, 4, 8, 12, 16, 20} $\subseteq$ S is a subsemigroup of S and o(M) / o(S).

W = {0, 8, 16} $\subseteq$ S is a subsemigroup such that o(W) / o(S). However S has subsemigroups H = {0, 12, 1, 6, 18} $\subseteq$ S is such that o(H) $\not|$ o(S). So S satisfies anti Lagrange semigroup property as well as S also satisfies the weak Lagrange's property.

Example 2.7: Let M = $\{Z_{12}, \times\}$ be the semigroup of order 12. B = {0, 6, 1, 3, 9} $\subseteq$ M is a subsemigroup of order 12. o(B) $\not|$ o(M). C = {0, 3, 6, 9} $\subseteq$ M is a subsemigroup of order 12. o(C) / o(M). Thus M satisfies both anti Lagrange property as well as weak Lagrange property.

In view of this the following proposition is important.

Proposition 2.2: Let S = $\{Z_n, \times\}$ be the semigroup n not a prime. S satisfies both anti Lagrange property and weak Lagrange's property.

Proof: Consider any ideal I of Z_n say of order m; the largest number that divides n; m/n. Consider P = I $\cup$ {1}, P is a subsemigroup of order m + 1 and then m +1 $\not|$ n; hence the claim.

This situation will be illustrated by the following example:

Example 2.8: Let S = $\{Z_{180}, \times\}$ be the semigroup.
$180 = 2^2 \times 3^2 \times 5$; to show S has subsemigroups of order.
2,4,3,9,5,10,20,45,15,18, 6 and 36.
Let H_1 = {0, 90}, H_2 = {0, 45, 90, 135}, H_3 = {0, 60, 120},
H_4 = {0, 20, 40, 60, 80,100,120, 140. 160},
H_5 = {0, 36, 72, 108, 144}, H_6 ={0, 18, 36, 54, 72, 90, 108, 126, 144, 162},
H_7 = {0, 9 18, 27, …, 171}, H_8 = {0, 4, 8, 12, 16, …, 176},
H_9 = {0, 12, 24, 36, …, 168}
H_{10}= {0, 10, 20, 30, …, 170}, H_{11} = {0,30, 60, 90,120, 150} and
H_{12} = {0, 5, 10, 15, 20, …, 175}
are all subsemigroups of order. 2,4,3,…, 36 respectively and all subsemigroups are such that their order divides order of S. Hence S satisfies the weak Lagrange's property.

In view of this one can say all semigroups S = $\{Z_n, \times$, n not a prime$\}$ are weak Lagrange semigroups.

Corollary 2.1: Let S = {Z_n, ×} n a prime. S does not satisfy weak Lagrange's property; but satisfies anti Lagrange property.

Next there is a class of semigroups known as the symmetric semigroup S(n) [3] and S(n) behaves in a very different way.

Some examples in this direction are given.

Example 2.9: Let S(3) be the symmetric group of degree there. S(3) has idempotents. S(3) satisfies both anti Lagrange's property as well as weak Lagrange's property.

For take

$$B_1 = \left\{\begin{pmatrix}1&2&3\\1&2&3\end{pmatrix},\begin{pmatrix}1&2&3\\2&3&1\end{pmatrix},\begin{pmatrix}1&2&3\\3&1&2\end{pmatrix}\right\} \subseteq S(3)$$

is a subgroup of order three. Let

$$B_2 = \left\{\begin{pmatrix}1&2&3\\1&2&3\end{pmatrix},\begin{pmatrix}1&2&3\\1&1&1\end{pmatrix}\right\} \quad S(3);$$

B_2 is a subsemigroup of order two. $o(B_1)$ / $o(S_3))$ as $o(S(3)) = 3^3$ and $o(B_2) \nmid o(S(3))$.

$$B_3 = \left\{\begin{pmatrix}1&2&3\\1&2&3\end{pmatrix},\begin{pmatrix}1&2&3\\1&1&1\end{pmatrix},\begin{pmatrix}1&2&3\\2&2&2\end{pmatrix}\right\} S(3);$$

is a subsemigroup of S(3) and $|B_3|=3$ and $3/3^3$.

Take

$$M = \left\{\begin{pmatrix}1&2&3\\1&2&3\end{pmatrix},\begin{pmatrix}1&2&3\\1&3&2\end{pmatrix}\right\} \subseteq S(3)$$

is a subgroup and $o(M) \nmid o(S(3))$. S(3) has subsemigroup of order 9.

For take

$$N=\left\{\begin{pmatrix}1&2&3\\1&2&3\end{pmatrix},\begin{pmatrix}1&2&3\\1&3&2\end{pmatrix},\begin{pmatrix}1&2&3\\3&2&1\end{pmatrix}\begin{pmatrix}1&2&3\\2&1&3\end{pmatrix},\begin{pmatrix}1&2&3\\3&1&2\end{pmatrix},\begin{pmatrix}1&2&3\\2&3&1\end{pmatrix},\right.$$
$$\left.\begin{pmatrix}1&2&3\\1&1&1\end{pmatrix},\begin{pmatrix}1&2&3\\2&2&2\end{pmatrix},\begin{pmatrix}1&2&3\\3&3&3\end{pmatrix}\right\}\subseteq S(3)$$

is a subsemigroup of order 9 and $9 / 3^3$. $\begin{pmatrix}1&2&3\\1&1&1\end{pmatrix}$, $\begin{pmatrix}1&2&3\\2&2&2\end{pmatrix}$ and $\begin{pmatrix}1&2&3\\3&3&3\end{pmatrix}$ are idempotents in S(3). For $\begin{pmatrix}1&2&3\\1&1&1\end{pmatrix} o \begin{pmatrix}1&2&3\\1&1&1\end{pmatrix} = \begin{pmatrix}1&2&3\\1&1&1\end{pmatrix}$.

Likewise for other two elements. Take

$$P_1 = \left\{\begin{pmatrix}1&2&3\\1&2&3\end{pmatrix},\begin{pmatrix}1&2&3\\1&1&1\end{pmatrix}\right\},$$

$$P_2 = \left\{\begin{pmatrix}1&2&3\\1&2&3\end{pmatrix},\begin{pmatrix}1&2&3\\2&2&2\end{pmatrix}\right\}$$

and

$$P_3 = \left\{\begin{pmatrix}1&2&3\\1&2&3\end{pmatrix},\begin{pmatrix}1&2&3\\3&3&3\end{pmatrix}\right\}$$

are subsemigroups of order two $o(P_i) \nmid o(S(3))$ for i = 1, 2, 3. Hence S(3) enjoys anti Lagrange's property also.

In view of this the following proposition is proved.

Proposition 2.3: Let S(n), be the symmetric semigroup (n odd). S(n) satisfies anti Lagrange's property.

Proof: Given n is odd so $o(S(n)) = n^n$ and $2 \nmid o(S(n))$.

Let

$$P_1=\left\{\begin{pmatrix}1&2&3&4&\dots&n\\1&2&3&4&\dots&n\end{pmatrix},\begin{pmatrix}1&2&3&4&\dots&n\\1&1&1&1&\dots&1\end{pmatrix}\right\}\subseteq S(n)$$

P_1 is a semigroup of order 2. $o(P_1) \nmid o(S(n))$ so S(n) satisfies the anti Lagrange's property.

Proposition 2.4: Let S(n) be the symmetric semigroup; n a even integer. S(n) satisfies the weak Lagrange's property.

Proof: $o(S(n)) = (n)^n$ (n an even integer). Thus 2 / o(S(n)).
Take

$$D_1=\left\{\begin{pmatrix}1&2&3&\cdots&n\\1&2&3&\dots&n\end{pmatrix},\begin{pmatrix}1&2&3&\cdots&n\\1&1&1&\dots&1\end{pmatrix}\right\}\subseteq S(n);$$

D_1 is a subsemigroup of order 2 and $o(D_1)$ / o(S(n)). Thus S(n) satisfies the weak Lagrange's property.

Theorem 2.1: Every symmetric semigroup S(n) satisfies anti Lagrange's property.

Proof: Let n be a prime; S(n) the symmetric semigroup of degree n. $o(S(n)) = n^n$; n a prime (n > 2). Let

$$P=\left\{\begin{pmatrix}1&2&3&\cdots&n\\1&2&3&\dots&n\end{pmatrix},\begin{pmatrix}1&2&3&\cdots&n\\1&1&1&\dots&1\end{pmatrix}\right\}\subseteq S(n)$$

be a subsemigroup such that $o(P) \nmid n^n$.

Hence S(n) satisfies the anti Lagrange's property.

Suppose n is not a prime consider the subsemigroup;

$$M=S_n\cup\left\{\begin{pmatrix}1&2&3&\cdots&n\\1&1&1&\dots&1\end{pmatrix},\begin{pmatrix}1&2&3&\cdots&n\\2&2&2&\dots&2\end{pmatrix},\begin{pmatrix}1&2&3&\cdots&n\\3&3&3&\dots&3\end{pmatrix},\dots\begin{pmatrix}1&2&3&\cdots&n\\n&n&n&\dots&n\end{pmatrix}\right\}\subseteq S(n)$$

is a subsemigroup of order n! + n = n ((n – 1)! + 1). Clearly $n((n-1)!+1) \nmid n^n$ if n is not a prime. Hence S(n) in this case also satisfies the anti Lagrange's property.

Theorem 2.2: Every symmetric semigroup S(n) satisfies the weak Lagrange property.

Proof. Let S(n) be the symmetric semigroup of order n^n. Take

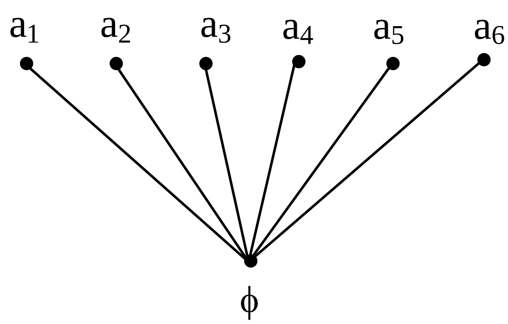
$$B=\left\{\begin{pmatrix}1&2&3&\cdots&n\\1&1&1&\dots&1\end{pmatrix},\begin{pmatrix}1&2&3&\cdots&n\\1&2&2&\dots&2\end{pmatrix},\begin{pmatrix}1&2&3&\cdots&n\\3&3&3&\dots&3\end{pmatrix},\dots,\begin{pmatrix}1&2&3&\cdots&n\\n&n&n&\dots&n\end{pmatrix}\right\}$$

$\subseteq$S(n) is a subsemigroup of order n and o(B) / o(S(n)). Thus the symmetric semigroup satisfies the weak Lagrange's property.

It is important to make mention that these two properties are not in any way related to the notion of Smarandache Lagrange's semigroup or Smarandache weakly Lagrange's semigroup defined in[3] for both are related to subgroups of a semigroup.

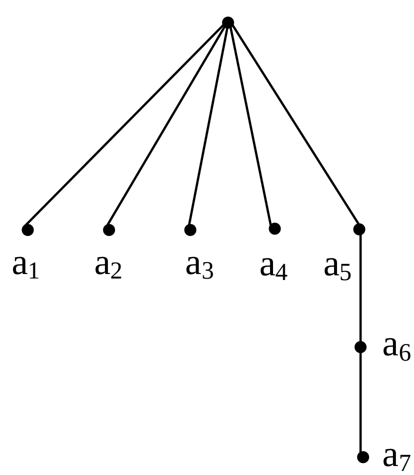
Finally the class of semilattices wither under $\cup$ or $\cap$ is an idempotent semigroup. Thus all finite semilattices are finite idempotent semigroups.

Example 2.10: Let S be the semilattice under the '∩' operation given by the following figure:



|S| = 7 and no subsemilattice (subsemigroup) H_i of S is such that $o(H_i)$ divides o(S). So this idempotent semigroup does not satisfy weak Lagrange's property for singleton elements of S are subsemigroups but they are considered as trivial subsemigroups.

Example 2.11: Consider the semilattice S under the binary operation ∪ given by the following figure:



$a_i \cup a_7 = 1$ for all i = 1, 2, 3 and 4, |S| = 8.

Take $P_1 = \{a_1, a_2, 1\} \subseteq S$ is a subsemigroup of order 3 and 3 ∤ 8. Thus S satisfies anti Lagrange's property. Let $P_2 = \{1\ a_5, a_6, a_7\} \subseteq S$ is a subsemigroup of order 4 and $o(P_2)$ / o(S). Thus S satisfies weak Lagrange's property.

Finally the following result is interesting.

Proposition 2.5: Let S be a semi lattice (idempotent semigroup) of order p; p a prime. S is only an idempotent semigroup which satisfies anti Lagrange's properties and S does not satisfy weak Lagrange property.

Proof follows from the fact order of S is a prime number.

3 Conclusions

In this paper study of non abstract finite semigroups which satisfy weak Lagrange's Property and anti Lagrange's property are analysed. Such study leads to visualize finite semigroups in par with finite groups and compare them.

Reference

1. Clifford, A.H. and Preston, G.B., *The algebraic theory of semigroups, Vol. I & II*, Mathematical Surveys of the Amer. Math. Soc. 7 (1961 & 1967).
2. Hall, M., *Theory of Groups*, Macmillian, (1961).
3. Vasantha,W.B, *Smarandache Semigroups, American Research Press,* (2002).