International J.Math. Combin. Vol.3(2014), 01-34

Mathematics on Non-Mathematics

— *A Combinatorial Contribution*

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Abstract: A classical system of mathematics is homogenous without contradictions. But it is a little ambiguous for modern mathematics, for instance, the Smarandache geometry. Let $\mathscr{F}$ be a family of things such as those of particles or organizations. Then, *how to hold its global behaviors or true face*? Generally, $\mathscr{F}$ is not a mathematical system in usual unless a set, i.e., a system with contradictions. There are no mathematical subfields applicable. Indeed, the trend of mathematical developing in 20th century shows that a mathematical system is more concise, its conclusion is more extended, but farther to the true face for its abandoned more characters of things. This effect implies an important step should be taken for mathematical development, i.e., turn the way to extending non-mathematics in classical to mathematics, which also be provided with the philosophy. All of us know *there always exists a universal connection between things in* $\mathscr{F}$. Thus there is an underlying structure, i.e., a vertex-edge labeled graph G for things in $\mathscr{F}$. Such a labeled graph G is invariant accompanied with $\mathscr{F}$. The main purpose of this paper is to survey how to extend classical mathematical non-systems, such as those of algebraic systems with contradictions, algebraic or differential equations with contradictions, geometries with contradictions, and generally, classical mathematics systems with contradictions to mathematics by the underlying structure G. All of these discussions show that a non-mathematics in classical is in fact a mathematics underlying a topological structure G, i.e., mathematical combinatorics, and contribute more to physics and other sciences.

Key Words: Non-mathematics, topological graph, Smarandache system, non-solvable equation, CC conjecture, mathematical combinatorics.

AMS(2010): 03A10,05C15,20A05, 34A26,35A01,51A05,51D20,53A35

§1. Introduction

A thing is complex, and hybrid with other things sometimes. That is why it is difficult to know the true face of all things, included in "Name named is not the eternal Name; the unnamable is the eternally real and naming the origin of all things", the first chapter of *TAO TEH KING* [9], a well-known Chinese book written by an ideologist, *Lao Zi* of China. In fact, all of things with

[1]Received February 16, 2014, Accepted August 8, 2014.

universal laws acknowledged come from the six organs of mankind. Thus, the words "*existence*" and "non-existence" are knowledged by human, which maybe not implies the true existence or not in the universe. Thus the existence or not for a thing is *invariant*, independent on human knowledge.

The boundedness of human beings brings about a unilateral knowledge for things in the world. Such as those shown in a famous proverb "the blind men with an elephant". In this proverb, there are six blind men were asked to determine what an elephant looked like by feeling different parts of the elephant's body. The man touched the elephant's leg, tail, trunk, ear, belly or tusk respectively claims it's like a pillar, a rope, a tree branch, a hand fan, a wall or a solid pipe, such as those shown in Fig.1 following. Each of them insisted on his own and not accepted others. They then entered into an endless argument.

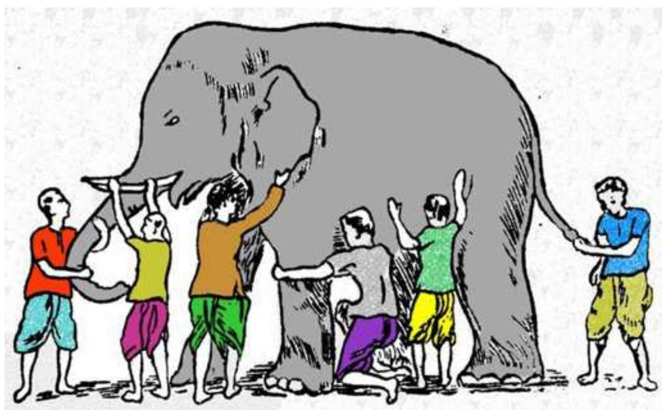


Fig.1

All of you are right! A wise man explains to them: *why are you telling it differently is because each one of you touched the different part of the elephant. So, actually the elephant has all those features what you all said*. Thus, the best result on an elephant for these blind men is

$$\begin{aligned}\text{An elephant} &= \{4\text{ pillars}\}\bigcup\{1\text{ rope}\}\bigcup\{1\text{ tree branch}\}\\ &\bigcup \{2\text{ hand fans}\}\bigcup\{1\text{ wall}\}\bigcup\{1\text{ solid pipe}\}\end{aligned}$$

What is the meaning of this proverb for understanding things in the world? It lies in that the situation of human beings knowing things in the world is analogous to these blind men. Usually, a thing T is identified with its known characters (or name) at one time, and this process is advanced gradually by ours. For example, let $\mu_1, \mu_2, \cdots, \mu_n$ be its known and $\nu_i, i \geq 1$ unknown characters at time t. Then, the thing T is understood by

$$T = \left(\bigcup_{i=1}^{n}\{\mu_i\}\right)\bigcup\left(\bigcup_{k\geq 1}\{\nu_k\}\right) \tag{1.1}$$

in logic and with an approximation $T^\circ = \bigcup\limits_{i=1}^{n}\{\mu_i\}$ for T at time t. This also answered why difficult for human beings knowing a thing really.

Generally, let Σ be a finite or infinite set. A *rule* or a *law* on a set Σ is a mapping $\underbrace{\Sigma\times\Sigma\cdots\times\Sigma}_{n}\rightarrow\Sigma$ for some integers n. Then, a *mathematical system* is a pair $(\Sigma;\mathcal{R})$, where $\mathcal{R}$ consists those of rules on Σ by logic providing all these resultants are still in Σ.

Definition 1.1([28]-[30]) *Let $(\Sigma_1;\mathcal{R}_1)$, $(\Sigma_2;\mathcal{R}_2)$, $\cdots$, $(\Sigma_m;\mathcal{R}_m)$ be m mathematical system, different two by two. A Smarandache multi-system $\widetilde{\Sigma}$ is a union $\bigcup\limits_{i=1}^{m}\Sigma_i$ with rules $\widetilde{\mathcal{R}}=\bigcup\limits_{i=1}^{m}\mathcal{R}_i$ on $\widetilde{\Sigma}$, denoted by $\left(\widetilde{\Sigma};\widetilde{\mathcal{R}}\right)$.*

Consequently, the thing T is nothing else but a Smarandache multi-system (1.1). However, these characters $\nu_k, k\geq 1$ are unknown for one at time t. Thus, $T\approx T^{\circ}$ is only an approximation for its true face and it will never be ended in this way for knowing T, i.e., "Name named is not the eternal Name", as Lao Zi said.

But one's life is limited by its nature. It is nearly impossible to find all characters $\nu_k,\ k\geq 1$ identifying with thing T. Thus one can only understands a thing T relatively, namely find invariant characters $\mathcal{I}$ on $\nu_k,\ k\geq 1$ independent on artificial frame of references. In fact, this notion is consistent with *Erlangen Programme* on developing geometry by Klein [10]: *given a manifold and a group of transformations of the same, to investigate the configurations belonging to the manifold with regard to such properties as are not altered by the transformations of the group*, also the fountainhead of *General Relativity* of Einstein [2]: *any equation describing the law of physics should have the same form in all reference frame*, which means that a universal law does not moves with the volition of human beings. Thus, an applicable mathematical theory for a thing T should be an *invariant theory* acting on by all automorphisms of the artificial frame of reference for thing T.

All of us have known that things are inherently related, not isolated in philosophy, which implies that these is an underlying structure in characters $\mu_i,\ 1\leq i\leq n$ for a thing T, namely, an inherited topological graph G. Such a graph G should be independent on the volition of human beings. Generally, a labeled graph G for a Smarandache multi-space is introduced following.

Definition 1.2([21]) *For any integer $m\geq 1$, let $\left(\widetilde{\Sigma};\widetilde{\mathcal{R}}\right)$ be a Smarandache multi-system consisting of m mathematical systems $(\Sigma_1;\mathcal{R}_1)$, $(\Sigma_2;\mathcal{R}_2)$, $\cdots$, $(\Sigma_m;\mathcal{R}_m)$. An inherited topological structure $G[\widetilde{S}]$ of $\left(\widetilde{\Sigma};\widetilde{\mathcal{R}}\right)$ is a topological vertex-edge labeled graph defined following:*

$$V(G[\widetilde{S}])=\{\Sigma_1,\Sigma_2,\cdots,\Sigma_m\},$$
$$E(G[\widetilde{S}])=\{(\Sigma_i,\Sigma_j)|\Sigma_i\bigcap\Sigma_j\neq\emptyset,\ 1\leq i\neq j\leq m\}\ \textit{with labeling}$$
$$L:\ \Sigma_i\rightarrow L(\Sigma_i)=\Sigma_i\quad\textit{and}\quad L:\ (\Sigma_i,\Sigma_j)\rightarrow L(\Sigma_i,\Sigma_j)=\Sigma_i\bigcap\Sigma_j$$

for integers $1\leq i\neq j\leq m$.

However, classical combinatorics paid attentions mainly on techniques for catering the need of other sciences, particularly, the computer science and children games by artificially giving up individual characters on each system $(\Sigma,\mathcal{R})$. For applying more it to other branch sciences initiatively, a good idea is pullback these individual characters on combinatorial objects again, ignored by the classical combinatorics, and back to the true face of things, i.e., an interesting conjecture on mathematics following:

Conjecture 1.3(CC Conjecture, [15],[19]) *A mathematics can be reconstructed from or turned into combinatorization.*

Certainly, this conjecture is true in philosophy. So it is in fact a combinatorial notion on developing mathematical sciences. Thus:

(1) *One can combine different branches into a new theory and this process ended until it has been done for all mathematical sciences, for instance, topological groups and Lie groups.*

(2) *One can selects finite combinatorial rulers and axioms to reconstruct or make generalizations for classical mathematics, for instance, complexes and surfaces.*

From its formulated, the CC conjecture brings about a new way for developing mathematics , and it has affected on mathematics more and more. For example, it contributed to groups, rings and modules ([11]-[14]), topology ([23]-[24]), geometry ([16]) and theoretical physics ([17]-[18]), particularly, these 3 monographs [19]-[21] motivated by this notion.

A *mathematical non-system* is such a system with contradictions. Formally, let $\mathscr{R}$ be mathematical rules on a set Σ. A pair $(\Sigma;\mathscr{R})$ is non-mathematics if there is at least one ruler $R \in \mathscr{R}$ validated and invalided on Σ simultaneously. Notice that a multi-system defined in Definition 1.1 is in fact a system with contradictions in the classical view, but it is cooperated with logic by Definition 1.2. Thus, it lights up the hope of transferring a system with contradictions to mathematics, consistent with logic by combinatorial notion.

The main purpose of this paper is to show how to transfer a mathematical non-system, such as those of non-algebra, non-group, non-ring, non-solvable algebraic equations, non-solvable ordinary differential equations, non-solvable partial differential equations and non-Euclidean geometry, mixed geometry, differential non- Euclidean geometry, $\cdots$, etc. classical mathematics systems with contradictions to mathematics underlying a topological structure G, i.e., mathematical combinatorics. All of these discussions show that *a mathematical non-system is a mathematical system inherited a non-trivial topological graph, respect to that of the classical underlying a trivial K_1 or K_2*. Applications of these non-mathematic systems to theoretical physics, such as those of gravitational field, infectious disease control, circulating economical field can be also found in this paper.

All terminologies and notations in this paper are standard. For those not mentioned here, we follow [1] and [19] for algebraic systems, [5] and [6] for algebraic invariant theory, [3] and [32] for differential equations, [4], [8] and [21] for topology and topological graphs and [20], [28]-[31] for Smarandache systems.

§2. Algebraic Systems

Notice that the graph constructed in Definition 1.2 is in fact on sets $\Sigma_i,\ 1 \leq i \leq m$ with relations on their intersections. Such combinatorial invariants are suitable for algebraic systems. All operations $\circ :\ \mathscr{A} \times \mathscr{A} \to \mathscr{A}$ on a set $\mathscr{A}$ considered in this section are closed and single valued, i.e., $a \circ b$ is uniquely determined in $\mathscr{A}$, and it is said to be *Abelian* if $a \circ b = b \circ a$ for

$\forall a, b \in \mathscr{A}$.

2.1 Non-Algebraic Systems

An algebraic system is a pair $(\mathscr{A};\mathcal{R})$ holds with $a \circ b \in \mathscr{A}$ for $\forall a, b \in \mathscr{A}$ and $\circ \in \mathcal{R}$. A *non-algebraic system* $\neg(\mathscr{A};\mathcal{R})$ on an algebraic system $(\mathscr{A};\mathcal{R})$ is

AS^{-1}: *there maybe exist an operation $\circ \in \mathcal{R}$, elements $a, b \in \mathscr{A}$ with $a \circ b$ undetermined.*

Similarly to classical algebra, an isomorphism on $\neg(\mathscr{A};\mathcal{R})$ is such a mapping on $\mathscr{A}$ that for $\forall \circ \in \mathcal{R}$,

$$h(a \circ b) = h(a) \circ h(b)$$

holds for $\forall a, b \in \mathscr{A}$ providing $a \circ b$ is defined in $\neg(\mathscr{A};\mathcal{R})$ and $h(a) = h(b)$ if and only if $a = b$. Not loss of generality, let $\circ \in \mathcal{R}$ be a chosen operation. Then, there exist closed subsets $\mathscr{C}_i,\ i \geq 1$ of $\mathscr{A}$. For instance,

$$\langle a \rangle^{\circ} = \{a, a \circ a, a \circ a \circ a, \cdots, \underbrace{a \circ a \circ \cdots \circ a}_{k}, \cdots\}$$

is a closed subset of $\mathscr{A}$ for $\forall a \in \mathscr{A}$. Thus, there exists a decomposition $\mathscr{A}_1^{\circ}, \mathscr{A}_2^{\circ}, \cdots, \mathscr{A}_n^{\circ}$ of $\mathscr{A}$ such that $a \circ b \in \mathscr{A}_i^{\circ}$ for $\forall a, b \in \mathscr{A}_i^{\circ}$ for integers $1 \leq i \leq n$.

Define a topological graph $G[\neg(\mathscr{A};\circ)]$ following:

$$V(G[\neg(\mathscr{A};\circ)]) = \{\mathscr{A}_1^{\circ}, \mathscr{A}_2^{\circ}, \cdots, \mathscr{A}_n^{\circ}\};$$
$$E(G[\neg(\mathscr{A};\circ)]) = \{(\mathscr{A}_i^{\circ}, \mathscr{A}_j^{\circ}) \text{ if } \mathscr{A}_i^{\circ} \bigcap \mathscr{A}_j^{\circ} \neq \emptyset, 1 \leq i, \neq j \leq n\}$$

with labels

$$L: \ \mathscr{A}_i^{\circ} \in V(G[\neg(\mathscr{A};\circ)]) \to L(\mathscr{A}_i^{\circ}) = \mathscr{A}_i^{\circ},$$
$$L: \ (\mathscr{A}_i^{\circ}, \mathscr{A}_j^{\circ}) \in E(G[\neg(\mathscr{A};\circ)]) \to \mathscr{A}_i^{\circ} \bigcap \mathscr{A}_j^{\circ} \text{ for integers } 1 \leq i \neq j \leq n.$$

For example, let $\mathscr{A}_1^{\circ} = \{a, b, c\}$, $\mathscr{A}_2^{\circ} = \{a, d, f\}$, $\mathscr{A}_3^{\circ} = \{c, d, e\}$, $\mathscr{A}_4^{\circ} = \{a, e, f\}$ and $\mathscr{A}_5^{\circ} = \{d, e, f\}$. Calculation shows that $\mathscr{A}_1^{\circ} \bigcap \mathscr{A}_2^{\circ} = \{a\}$, $\mathscr{A}_1^{\circ} \bigcap \mathscr{A}_3^{\circ} = \{c\}$, $\mathscr{A}_1^{\circ} \bigcap \mathscr{A}_4^{\circ} = \{a\}$, $\mathscr{A}_1^{\circ} \bigcap \mathscr{A}_5^{\circ} = \emptyset$, $\mathscr{A}_2^{\circ} \bigcap \mathscr{A}_3^{\circ} = \{d\}$, $\mathscr{A}_2^{\circ} \bigcap \mathscr{A}_4^{\circ} = \{a\}$, $\mathscr{A}_2^{\circ} \bigcap \mathscr{A}_5^{\circ} = \{d, f\}$,$\mathscr{A}_3^{\circ} \bigcap \mathscr{A}_4^{\circ} = \{e\}$, $\mathscr{A}_3^{\circ} \bigcap \mathscr{A}_5^{\circ} = \{d, e\}$ and $\mathscr{A}_4^{\circ} \bigcap \mathscr{A}_5^{\circ} = \{e, f\}$. Then, the labeled graph $G[\neg(\mathscr{A};\circ)]$ is shown in Fig.2.

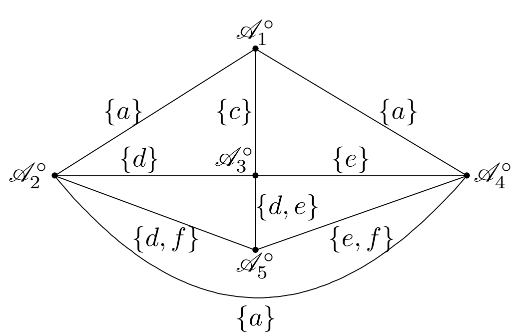


Fig.2

Let $h: \mathscr{A} \rightarrow \mathscr{A}$ be an isomorphism on $\neg(\mathscr{A};\circ)$. Then $\forall a,b \in \mathscr{A}_i^\circ)$, $h(a)\circ h(b) = h(a\circ b) \in h(A_i^\circ)$ and $h(A_i^\circ)\bigcap h(A_j^\circ) = h(A_i^\circ \bigcap A_j^\circ) = \emptyset$ if and only if $A_i^\circ \bigcap A_j^\circ = \emptyset$ for integers $1 \leq i \neq j \leq n$. Whence, if $G^h[\neg(\mathscr{A};\circ)]$ defined by

$$V(G^h[\neg(\mathscr{A};\circ)]) = \{h(\mathscr{A}_1^\circ), h(\mathscr{A}_2^\circ), \cdots, h(\mathscr{A}_n^\circ)\};$$
$$E(G^h[\neg(\mathscr{A};\circ)]) = \{(h(\mathscr{A}_i^\circ), h(\mathscr{A}_j^\circ)) \text{ if } h(\mathscr{A}_i^\circ)\bigcap h(\mathscr{A}_j^\circ) \neq \emptyset, 1 \leq i, \neq j \leq n\}$$

with labels

$$L^h: \ h(\mathscr{A}_i^\circ) \in V(G^h[\neg(\mathscr{A};\circ)]) \rightarrow L(h(\mathscr{A}_i^\circ)) = h(\mathscr{A}_i^\circ),$$
$$L^h: \ (h(\mathscr{A}_i^\circ), h(\mathscr{A}_j^\circ)) \in E(G^h[\neg(\mathscr{A};\circ)]) \rightarrow h(\mathscr{A}_i^\circ)\bigcap h(\mathscr{A}_j^\circ)$$

for integers $1 \leq i \neq j \leq n$. Thus $h: \mathscr{A} \rightarrow \mathscr{A}$ induces an isomorphism of graph $h^*: G[\neg(\mathscr{A};\circ)] \rightarrow G^h[\neg(\mathscr{A};\circ)]$. We therefore get the following result.

Theorem 2.1 *A non-algebraic system $\neg(\mathscr{A};\circ)$ in type AS^{-1} inherits an invariant $G[\neg(\mathscr{A};\circ)]$ of labeled graph.*

Let

$$G[\neg(\mathscr{A};\mathcal{R})] = \bigcup_{\circ\in\mathcal{R}} G[\neg(\mathscr{A};\circ)]$$

be a topological graph on $\neg(\mathscr{A};\mathcal{R})$. Theorem 2.1 naturally leads to the conclusion for non-algebraic system $\neg(\mathscr{A};\mathcal{R})$ following.

Theorem 2.2 *A non-algebraic system $\neg(\mathscr{A};\mathcal{R})$ in type AS^{-1} inherits an invariant $G[\neg(\mathscr{A};\mathcal{R})]$ of topological graph.*

Similarly, we can also discuss *algebraic non-associative systems*, *algebraic non-Abelian systems* and find inherited invariants $G[\neg(\mathscr{A};\circ)]$ of graphs. Usually, we adopt different notations for operations in $\mathcal{R}$, which consists of a multi-system $(\mathscr{A};\mathcal{R})$. For example, $\mathcal{R} = \{+,\cdot\}$ in an algebraic field $(R;+,\cdot)$. If we view the operation $+$ is the same as $\cdot$, throw out $0\cdot a$, $a\cdot 0$ and $1+a$, $a+1$ for $\forall a \in R$ in R, then $(R;+,\cdot)$ comes to be a non-algebraic system $(R;\cdot)$ with topological graph $G[R;\cdot]$ shown in Fig.3.

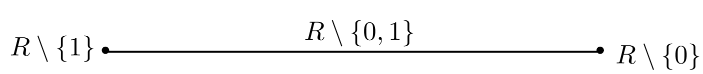


Fig.3

2.2 Non-Groups

A group is an associative system $(\mathscr{G};\circ)$ holds with identity and inverse elements for all elements in $\mathscr{G}$. Thus, for $a,b,c \in \mathscr{G}$, $(a\circ b)\circ c = a\circ(b\circ c)$, $\exists 1_{\mathscr{G}} \in \mathscr{G}$ such that $1_{\mathscr{G}}\circ a = a\circ 1_{\mathscr{G}} = a$ and for $\forall a \in \mathscr{G}$, $\exists a^{-1} \in \mathscr{A}\mathscr{G}$ such that $a\circ a^{-1} = 1_{\mathscr{G}}$. A *non-group* $\neg(\mathscr{G};\circ)$ on a group $(\mathscr{G};\circ)$ is an algebraic system in 3 types following:

AG_1^{-1}: *there maybe exist a_1, b_1, c_1 and $a_2, b_2, c_2 \in \mathscr{G}$ such that $(a_1 \circ b_1) \circ c_1 = a_1 \circ (b_1 \circ c_1)$ but $(a_2 \circ b_2) \circ c_2 \neq a_2 \circ (b_2 \circ c_2)$, also holds with identity $1_{\mathscr{G}}$ and inverse element a^{-1} for all elements in $a \in \mathscr{G}$.*

AG_2^{-1}: *there maybe exist distinct $1_{\mathscr{G}}, 1'_{\mathscr{G}} \in \mathscr{G}$ such that $a_1 \circ 1_{\mathscr{G}} = 1_{\mathscr{G}} \circ a_1 = a_1$ and $a_2 \circ 1'_{\mathscr{G}} = 1'_{\mathscr{G}} \circ a_2 = a_2$ for $a_1 \neq a_2 \in \mathscr{G}$, also holds with associative and inverse elements a^{-1} on $1_{\mathscr{G}}$ and $1'_{\mathscr{G}}$ for $\forall a \in \mathscr{G}$.*

AG_3^{-1}: *there maybe exist distinct inverse elements $a^{-1}, \dot{a}^{-1}$ for $a \in \mathscr{G}$, also holds with associative and identity elements.*

Notice that $(a \circ a) \circ a = a \circ (a \circ a)$ always holds with $a \in \mathscr{G}$ in an algebraic system. Thus there exists a decomposition $\mathscr{G}_1, \mathscr{G}_2, \cdots, \mathscr{G}_n$ of $\mathscr{G}$ such that $(\mathscr{G}_i; \circ)$ is a group for integers $1 \leq i \leq n$ for Type AG_1^{-1}.

Type AG_2^{-1} is true only if $1_{\mathscr{G}} \circ 1'_{\mathscr{G}} \neq 1_{\mathscr{G}}$ and $\neq 1'_{\mathscr{G}}$. Thus $1_{\mathscr{G}}$ and $1'_{\mathscr{G}}$ are local, not a global identity on $\mathscr{G}$. Define

$$\mathscr{G}(1_{\mathscr{G}}) = \{a \in \mathscr{G} \text{ if } a \circ 1_{\mathscr{G}} = 1_{\mathscr{G}} \circ a = a\}.$$

Then $\mathscr{G}(1_{\mathscr{G}}) \neq \mathscr{G}(1'_{\mathscr{G}})$ if $1_{\mathscr{G}} \neq 1'_{\mathscr{G}}$. Denoted by $I(\mathscr{G})$ the set of all local identities on $\mathscr{G}$. Then $\mathscr{G}(1_{\mathscr{G}}),\ 1_{\mathscr{G}} \in I(\mathscr{G})$ is a decomposition of $\mathscr{G}$ such that $(\mathscr{G}(1_{\mathscr{G}}); \circ)$ is a group for $\forall 1_{\mathscr{G}} \in I(\mathscr{G})$.

Type AG_3^{-1} is true only if there are distinct local identities $1_{\mathscr{G}}$ on $\mathscr{G}$. Denoted by $I(\mathscr{G})$ the set of all local identities on $\mathscr{G}$. We can similarly find a decomposition of $\mathscr{G}$ with group $(\mathscr{G}(1_{\mathscr{G}}); \circ)$ holds for $\forall 1_{\mathscr{G}} \in I(\mathscr{G})$ in this type.

Thus, for a non-group $\neg(\mathscr{G}; \circ)$ of AG_1^{-1}-AG_3^{-1}, we can always find groups $(\mathscr{G}_1; \circ), (\mathscr{G}_2; \circ), \cdots$, $(\mathscr{G}_n; \circ)$ for an integer $n \geq 1$ with $\mathscr{G} = \bigcup\limits_{i=1}^{n} \mathscr{G}_i$. Particularly, if $(\mathscr{G}; \circ)$ is itself a group, then such a decomposition is clearly exists by its subgroups.

Define a topological graph $G[\neg(\mathscr{G}; \circ)]$ following:

$$V(G[\neg(\mathscr{G}; \circ)]) = \{\mathscr{G}_1, \mathscr{G}_2, \cdots, \mathscr{G}_n\};$$
$$E(G[\neg(\mathscr{G}; \circ)]) = \{(\mathscr{G}_i, \mathscr{G}_j) \text{ if } \mathscr{G}_i \bigcap \mathscr{G}_j \neq \emptyset, 1 \leq i, \neq j \leq n\}$$

with labels

$$L:\ \mathscr{G}_i \in V(G[\neg(\mathscr{G}; \circ)]) \to L(\mathscr{G}_i) = \mathscr{G}_i,$$
$$L:\ (\mathscr{G}_i, \mathscr{G}_j) \in E(G[\neg(\mathscr{G}; \circ)]) \to \mathscr{G}_i \bigcap \mathscr{G}_j \text{ for integers } 1 \leq i \neq j \leq n.$$

For example, let $\mathscr{G}_1 = \langle \alpha, \beta \rangle$, $\mathscr{G}_2 = \langle \alpha, \gamma, \theta \rangle$, $\mathscr{G}_3 = \langle \beta, \gamma \rangle$, $\mathscr{G}_4 = \langle \beta, \delta, \theta \rangle$ be 4 free Abelian groups with $\alpha \neq \beta \neq \gamma \neq \delta \neq \theta$. Calculation shows that $\mathscr{G}_1 \bigcap \mathscr{G}_2 = \langle \alpha \rangle$, $\mathscr{G}_2 \bigcap \mathscr{G}_3 = \langle \gamma \rangle$, $\mathscr{G}_3 \bigcap \mathscr{G}_4 = \langle \delta \rangle$, $\mathscr{G}_1 \bigcap \mathscr{G}_4 = \langle \beta \rangle$ and $\mathscr{G}_2 \bigcap \mathscr{G}_4 = \langle \theta \rangle$. Then, the topological graph $G[\neg(\mathscr{G}; \circ)]$ is shown in Fig.4.

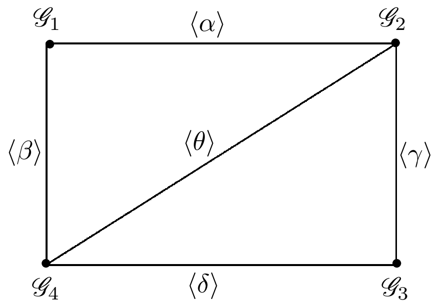


Fig.4

For an isomorphism $g: \mathscr{G} \to \mathscr{G}$ on $\neg(\mathscr{G};\circ)$, it naturally induces a 1-1 mapping $g^*: V(G[\neg(\mathscr{G};\circ)]) \to V(G[\neg(\mathscr{G};\circ)])$ such that each $g^*(\mathscr{G}_i)$ is also a group and $g^*(\mathscr{G}_i)\bigcap g^*(\mathscr{G}_j) \neq \emptyset$ if and only if $\mathscr{G}_i\bigcap\mathscr{G}_j \neq \emptyset$ for integers $1 \leq i \neq j \leq n$. Thus g induced an isomorphism g^* of graph from $G[\neg(\mathscr{G};\circ)]$ to $g^*(G[\neg(\mathscr{G};\circ)])$, which implies a conclusion following.

Theorem 2.3 *A non-group $\neg(\mathscr{G};\circ)$ in type AG_1^{-1}-AG_3^{-1} inherits an invariant $G[\neg(\mathscr{G};\circ)]$ of topological graph.*

Similarly, we can discuss more non-groups with some special properties, such as those of *non-Abelian group*, *non-solvable group*, *non-nilpotent group* and find inherited invariants $G[\neg(\mathscr{G};\circ)]$. Notice that([19]) any group $\mathscr{G}$ can be decomposed into disjoint classes $C(H_1)$, $C(H_2), \cdots, C(H_s)$ of conjugate subgroups, particularly, disjoint classes $Z(a_1)$, $Z(a_2), \cdots, Z(a_l)$ of centralizers with $|C(H_i)| = |\mathscr{G} : N_{\mathscr{G}}(H_i)|$, $|Z(a_j)| = |\mathscr{G} : Z_{\mathscr{G}}(a_j)|$, $1 \leq i \leq s$, $1 \leq j \leq l$ and $|C(H_1)| + |C(H_2)| + \cdots + |C(H_s)| = |\mathscr{G}|$, $|Z(a_1)| + |Z(a_2)| + \cdots + |Z(a_l)| = |\mathscr{G}|$, where $N_{\mathscr{G}}(H)$, $Z(a)$ denote respectively the normalizer of subgroup H and centralizer of element a in group $\mathscr{G}$. This fact enables one furthermore to construct topological structures of non-groups with special classes of groups following:

Replace a vertex $\mathscr{G}_i$ by s_i (or l_i) isolated vertices labeled with $C(H_1)$, $C(H_2), \cdots, C(H_{s_i})$ (or $Z(a_1)$, $Z(a_2), \cdots, Z(a_{l_i})$) in $G[\neg(\mathscr{G};\circ)]$ and denoted the resultant by $\widehat{G}[\neg(\mathscr{G};\circ)]$.

We then get results following on non-groups with special topological structures by Theorem 2.3.

Theorem 2.4 *A non-group $\neg(\mathscr{G};\circ)$ in type AG_1^{-1}-AG_3^{-1} inherits an invariant $\widehat{G}[\neg(\mathscr{G};\circ)]$ of topological graph labeled with conjugate classes of subgroups on its vertices.*

Theorem 2.5 *A non-group $\neg(\mathscr{G};\circ)$ in type AG_1^{-1}-AG_3^{-1} inherits an invariant $\widehat{G}[\neg(\mathscr{G};\circ)]$ of topological graph labeled with Abelian subgroups, particularly, with centralizers of elements in $\mathscr{G}$ on its vertices.*

Particularly, for a group the following is a readily conclusion of Theorems 2.4 and 2.5.

Corollary 2.6 *A group $(\mathscr{G};\circ)$ inherits an invariant $\widehat{G}[\mathscr{G};\circ]$ of topological graph labeled with conjugate classes of subgroups (or centralizers) on its vertices, with $E(\widehat{G}[\mathscr{G};\circ]) = \emptyset$*

2.3 Non-Rings

A ring is an associative algebraic system $(R;+,\circ)$ on 2 binary operations "+", "$\circ$", hold with an Abelian group $(R;+)$ and for $\forall x,y,z\in R$, $x\circ(y+z)=x\circ y+x\circ z$ and $(x+y)\circ z=x\circ z+y\circ z$. Denote the identity by 0_+, the inverse of a by $-a$ in $(R;+)$. A *non-ring* $\neg(R;+,\circ)$ on a ring $(R;+,\circ)$ is an algebraic system on operations "+", "$\circ$" in 5 types following:

$\mathbf{AR}_1^{-1}$: there maybe exist $a,b\in R$ such that $a+b\neq b+a$, but hold with the associative in $(R;\circ)$ and a group $(R;+)$;

$\mathbf{AR}_2^{-1}$: there maybe exist a_1,b_1,c_1 and $a_2,b_2,c_2\in R$ such that $(a_1\circ b_1)\circ c_1=a_1\circ(b_1\circ c_1)$, $(a_2\circ b_2)\circ c_2\neq a_2\circ(b_2\circ c_2)$, but holds with an Abelian group $(R;+)$.

$\mathbf{AR}_3^{-1}$: there maybe exist a_1,b_1,c_1 and $a_2,b_2,c_2\in R$ such that $(a_1+b_1)+c_1=a_1+(b_1+c_1)$, $(a_2+b_2)+c_2\neq a_2+(b_2+c_2)$, but holds with $(a\circ b)\circ c=a\circ(b\circ c)$, identity 0_+ and $-a$ in $(R;+)$ for $\forall a,b,c\in R$.

$\mathbf{AR}_4^{-1}$: there maybe exist distinct $0_+,0'_+\in R$ such that $a+0_+=0_++a=a$ and $b+0'_+=0'_++b=b$ for $a\neq b\in R$, but holds with the associative in $(R;+)$, $(R;\circ)$ and inverse elements $-a$ on 0_+, $0'_+$ in $(R;+)$ for $\forall a\in R$.

$\mathbf{AR}_5^{-1}$: there maybe exist distinct inverse elements $-a,-\dot{a}$ for $a\in R$ in $(R;+)$, but holds with the associative in $(R;+)$, $(R;\circ)$ and identity elements in $(R;+)$.

Notice that $(a+a)+a=a+(a+a)$, $a+a=a+a$ and $a\circ a=a\circ$ always hold in non-ring $\neg(R;+,\circ)$. Whence, for Types AR_1^{-1} and AR_2^{-1}, there exists a decomposition $R_1,R_2,\cdots,R_n$ of R such that $a+b=b+a$ and $(a\circ b)\circ c=a\circ(b\circ c)$ if $a,b,c\in R_i$, i.e., each $(R_i;+,\circ)$ is a ring for integers $1\leq i\leq n$. A similar discussion for Types AG_1^{-1}-AG_3^{-1} in Section 2.2 also shows such a decomposition $(R_i;+,\circ)$, $1\leq i\leq n$ of subrings exists for Types $3-5$. Define a topological graph $G[\neg(R;+,\circ)]$ by

$$V(G[\neg(R;+,\circ)])=\{R_1,R_2,\cdots,R_n\};$$
$$E(G[\neg(R;+,\circ)])=\{(R_i,R_j)\text{ if }R_i\bigcap R_j\neq\emptyset,1\leq i,\neq j\leq n\}$$

with labels

$$L:\ R_i\in V(G[\neg(R;+,\circ)])\rightarrow L(R_i)=R_i,$$
$$L:\ (R_i,R_j)\in E(G[\neg(R;+,\circ)])\rightarrow R_i\bigcap R_j\text{ for integers }1\leq i\neq j\leq n.$$

Then, such a topological graph $G[\neg(R;+,\circ)]$ is also an invariant under isomorphic actions on $\neg(R;+,\circ)$. Thus,

Theorem 2.7 *A non-ring $\neg(R;+,\circ)$ in types AR_1^{-1}-AR_5^{-1} inherits an invariant $G[\neg(R;+,\circ)]$ of topological graph.*

Furthermore, we can consider *non-associative ring*, *non-integral domain*, *non-division ring*, *skew non-field* or *non-field*, $\cdots$, etc. and find inherited invariants $G[\neg(R;+,\circ)]$ of graphs. For example, a *non-field* $\neg(F;+,\circ)$ on a field $(F;+,\circ)$ is an algebraic system on operations "+", "$\circ$" in 8 types following:

$\mathbf{AF}_1^{-1}$: there maybe exist a_1, b_1, c_1 and $a_2, b_2, c_2 \in F$ such that $(a_1 \circ b_1) \circ c_1 = a_1 \circ (b_1 \circ c_1)$, $(a_2 \circ b_2) \circ c_2 \neq a_2 \circ (b_2 \circ c_2)$, but holds with an Abelian group $(F;+)$, identity $1_\circ$, a^{-1} for $a \in F$ in $(F;\circ)$.

$\mathbf{AF}_2^{-1}$: there maybe exist a_1, b_1, c_1 and $a_2, b_2, c_2 \in F$ such that $(a_1+b_1)+c_1 = a_1+(b_1+c_1)$, $(a_2 + b_2) + c_2 \neq a_2 + (b_2 + c_2)$, but holds with an Abelian group $(F;\circ)$, identity 1_+, $-a$ for $a \in F$ in $(F;+)$.

$\mathbf{AF}_3^{-1}$: there maybe exist $a, b \in F$ such that $a \circ b \neq b \circ a$, but hold with an Abelian group $(F;+)$, a group $(F;\circ)$;

$\mathbf{AF}_4^{-1}$: there maybe exist $a, b \in F$ such that $a + b \neq b + a$, but hold with a group $(F;+)$, an Abelian group $(F;\circ)$;

$\mathbf{AF}_5^{-1}$: there maybe exist distinct $0_+, 0'_+ \in F$ such that $a + 0_+ = 0_+ + a = a$ and $b + 0'_+ = 0'_+ + b = b$ for $a \neq b \in F$, but holds with the associative, inverse elements $-a$ on 0_+, $0'_+$ in $(F;+)$ for $\forall a \in F$, an Abelian group $(F;\circ)$;

$\mathbf{AF}_6^{-1}$: there maybe exist distinct $1_\circ, 1'_\circ \in F$ such that $a \circ 1_\circ = 1_\circ \circ a = a$ and $b \circ 1'_\circ = 1'_\circ \circ b = b$ for $a \neq b \in F$, but holds with the associative, inverse elements a^{-1} on $1_\circ$, $1'_\circ$ in $(F;\circ)$ for $\forall a \in F$, an Abelian group $(F;+)$;

$\mathbf{AF}_7^{-1}$: there maybe exist distinct inverse elements $-a, -\dot{a}$ for $a \in F$ in $(F;+)$, but holds with the associative, identity elements in $(F;+)$, an Abelian group $(F;\circ)$.

$\mathbf{AF}_8^{-1}$: there maybe exist distinct inverse elements $a^{-1}, \dot{a}^{-1}$ for $a \in F$ in $(F;\circ)$, but holds with the associative, identity elements in $(F;\circ)$, an Abelian group $(F;+)$.

Similarly, we can show that there exists a decomposition $(F_i;+,\circ)$, $1 \leq i \leq n$ of fields for non-fields $\neg(F;+,\circ)$ in Types AF_1^{-1}-AF_8^{-1} and find an invariant $G[\neg(F;+,\circ)]$ of graph.

Theorem 2.8 *A non-ring $\neg(F;+,\circ)$ in types AF_1^{-1}-AF_8^{-1} inherits an invariant $G[\neg(F;+,\circ)]$ of topological graph.*

2.4 Algebraic Combinatorics

All of previous discussions with results in Sections 2.1-2.3 lead to a conclusion alluded in philosophy that *a non-algebraic system $\neg(\mathscr{A};\mathcal{R})$ constraint with property can be decomposed into algebraic systems with the same constraints, and inherits an invariant $G[\neg(\mathscr{A};\mathcal{R})]$ of topological graph labeled with those of algebraic systems*, i.e., algebraic combinatorics, which is in accordance with the notion for developing geometry that of Klein's. Thus, a more applicable approach for developing algebra is including non-algebra to algebra by consider various non-algebraic systems constraint with property, but such a process will never be ended if we do not firstly determine all algebraic systems. Even though, a more feasible approach is by its inverse, i.e., algebraic G-systems following:

Definition 2.9 *Let $(\mathscr{A}_1;\mathcal{R}_1), (\mathscr{A}_2;\mathcal{R}_2), \cdots, (\mathscr{A}_n;\mathcal{R}_n)$ be algebraic systems. An algebraic G-system is a topological graph G with labeling $L:\ v \in V(G) \to L(v) \in \{\mathscr{A}_1, \mathscr{A}_2, \cdots, \mathscr{A}_n\}$ and $L:\ (u,v) \in E(G) \to L(u) \bigcap L(v)$ with $L(u) \bigcap L(v) \neq \emptyset$, denoted by $G[\mathscr{A},\mathcal{R}]$, where $\mathscr{A} = \bigcup\limits_{i=1}^{n} \mathscr{A}_i$*

and $\mathcal{R} = \bigcup_{i=1}^{n} \mathcal{R}_i$.

Clearly, if $G[\mathscr{A}, \mathcal{R}]$ is prescribed, these algebraic systems $(\mathscr{A}_1; \mathcal{R}_1), (\mathscr{A}_2; \mathcal{R}_2), \cdots, (\mathscr{A}_n; \mathcal{R}_n)$ with intersections are determined.

Problem 2.10 *Characterize algebraic G-systems $G[\mathscr{A}, \mathcal{R}]$, such as those of G-groups, G-rings, integral G-domain, skew G-fields, G-fields, $\cdots$, etc., or their combination $G - \{groups, rings\}$, $G - \{groups, integral\ domains\}$, $G - \{groups, fields\}$, $G - \{rings, fields\}$ $\cdots$. Particularly, characterize these G-algebraic systems for complete graphs $G = K_2, K_3, K_4$, path P_3, P_4 or circuit C_4 of order≤ 4.*

In this perspective, classical algebraic systems are nothing else but mostly algebraic K_1-systems, also a few algebraic K_2-systems. For example, a field $(F; +, \cdot)$ is in fact a K_2-group prescribed by Fig.3.

§3. Algebraic Equations

All equations discussed in this paper are independent, maybe contain one or several unknowns, not an impossible equality in algebra, for instance $2^{x+y+z} = 0$.

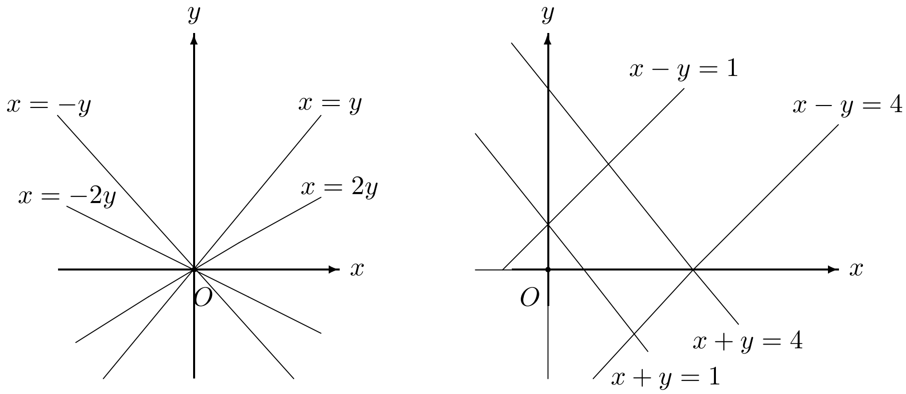
3.1 Geometry on Non-Solvable Equations

Let (LES_4^1), (LES_4^2) be two systems of linear equations following:

$$(LES_4^1) \quad \begin{cases} x = y \\ x = -y \\ x = 2y \\ x = -2y \end{cases} \qquad\qquad (LES_4^2) \quad \begin{cases} x + y = 1 \\ x + y = 4 \\ x - y = 1 \\ x - y = 4 \end{cases}$$

Clearly, the system (LES_4^1) is solvable with $x = 0, y = 0$ but (LES_4^2) is non-solvable because $x + y = 1$ is contradicts to that of $x + y = 4$ and so for $x - y = 1$ to $x - y = 4$. Even so, *is the system (LES_4^2) meaningless in the world*? Similarly, *is only the solution $x = 0,\ y = 0$ of system (LES_4^1) important to one*? Certainly NOT! This view can be readily come into being by all figures on $\mathbb{R}^2$ of these equations shown in Fig.5. Thus, if we denote by

$$\left\{ \begin{array}{l} L_1 = \{(x,y) \in \mathbb{R}^2 | x = y\} \\ L_2 = \{(x,y) \in \mathbb{R}^2 | x = -y\} \\ L_3 = \{(x,y) \in \mathbb{R}^2 | x = 2y\} \\ L_4 = \{(x,y) \in \mathbb{R}^2 | x = -2y\} \end{array} \right\} \quad \text{and} \quad \left\{ \begin{array}{l} L_1' = \{(x,y) \in \mathbb{R}^2 | x + y = 1\} \\ L_2' = \{(x,y) \in \mathbb{R}^2 | x + y = 4\} \\ L_3' = \{(x,y) \in \mathbb{R}^2 | x - y = 1\} \\ L_4' = \{(x,y) \in \mathbb{R}^2 | x - y = 4\} \end{array} \right\},$$



(LES_4^1) (LES_4^2)

Fig.5

the global behavior of (LES_4^1), (LES_4^2) are lines $L1 - L_4$, lines $L_1' - L'4$ on $\mathbb{R}^2$ and

$$L_1 \bigcap L_2 \bigcap L_3 \bigcap L_4 = \{(0,0)\} \text{ but } L_1' \bigcap L_2' \bigcap L_3' \bigcap L_4' = \emptyset.$$

Generally, let

$$(ES_m) \quad \begin{cases} f_1(x_1, x_2, \cdots, x_n) = 0 \\ f_2(x_1, x_2, \cdots, x_n) = 0 \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ f_m(x_1, x_2, \cdots, x_n) = 0 \end{cases}$$

be a system of algebraic equations in Euclidean space $\mathbb{R}^n$ for integers $m, n \geq 1$ with non-empty point set $S_{f_i} \subset \mathbb{R}^n$ such that $f_i(x_1, x_2, \cdots, x_n) = 0$ for $(x_1, x_2, \cdots, x_n) \in S_{f_i}$, $1 \leq i \leq m$. Clearly, the system (ES_m) is non-solvable or not dependent on

$$\bigcap_{i=1}^{m} S_{f_i} = \emptyset \text{ or } \neq \emptyset.$$

Conversely, let $\mathscr{G}$ be a geometrical space consisting of m parts $\mathscr{G}_1, \mathscr{G}_2, \cdots, \mathscr{G}_m$ in $\mathbb{R}^n$, where, each $\mathscr{G}_i$ is determined by a system of algebraic equations

$$\begin{cases} f_1^{[i]}(x_1, x_2, \cdots, x_n) = 0 \\ f_2^{[i]}(x_1, x_2, \cdots, x_n) = 0 \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ f_{m_i}^{[i]}(x_1, x_2, \cdots, x_n) = 0 \end{cases}$$

Then, the system of equations

$$\left.\begin{array}{l} f_1^{[i]}(x_1, x_2, \cdots, x_n) = 0 \\ f_2^{[i]}(x_1, x_2, \cdots, x_n) = 0 \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ f_{m_i}^{[i]}(x_1, x_2, \cdots, x_n) = 0 \end{array}\right\} \quad 1 \leq i \leq m$$

is non-solvable or not dependent on

$$\bigcap_{i=1}^{m}\mathscr{G}_i=\emptyset \quad \text{or} \quad \neq\emptyset.$$

Thus we obtain the following result.

Theorem 3.1 *The geometrical figure of equation system (ES_m) is a space $\mathscr{G}$ consisting of m parts $\mathscr{G}_i$ determined by equation $f_i(x_1,x_2,\cdots,x_n)=0,\ 1\leq i\leq m$ in (ES_m), and is non-solvable if $\bigcap\limits_{i=1}^{m}\mathscr{G}_i=\emptyset$. Conversely, if a geometrical space $\mathscr{G}$ consisting of m parts,$\mathscr{G}_1,\mathscr{G}_2,\cdots,\mathscr{G}_m$, each of them is determined by a system of algebraic equations in $\mathbb{R}^n$, then all of these equations consist a system (ES_m), which is non-solvable or not dependent on $\bigcap\limits_{i=1}^{m}\mathscr{G}_i=\emptyset$ or not.*

For example, let G be a planar graph with vertices v_1,v_2,v_3,v_4 and edges v_1v_2,v_1v_3,v_2v_3, v_3v_4, v_4v_1, shown in Fig.6.

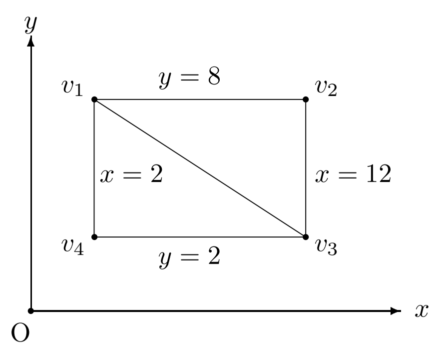


Fig.6

Then, a non-solvable system of equations with figure G on $\mathbb{R}^2$ consists of

$$(LE_5)\begin{cases} x=2 \\ y=8 \\ x=12 \\ y=2 \\ 3x+5y=46. \end{cases}$$

Thus G is an underlying graph of non-solvable system (LE_5).

Definition 3.2 *Let (ES_{m_i}) be a solvable system of m_i equations*

$$\begin{cases} f_1^{[i]}(x_1,x_2,\cdots,x_n)=0 \\ f_2^{[i]}(x_1,x_2,\cdots,x_n)=0 \\ \dots\dots\dots\dots\dots\dots\dots \\ f_{m_i}^{[i]}(x_1,x_2,\cdots,x_n)=0 \end{cases}$$

with a solution space $S_{f[i]}$ in $\mathbb{R}^n$ for integers $1 \leq i \leq m$. A topological graph $G[ES_m]$ is defined by

$$V(G[ES_m]) = \{S_{f[i]},\ 1 \leq i \leq m\};$$
$$E(G[ES_m]) = \{(S_{f[i]}, S_{f[j]}) \text{ if } S_{f[i]} \bigcap S_{f[j]} \neq \emptyset,\ 1 \leq i \neq j \leq m\}$$

with labels

$$L:\ S_{f[i]} \in V(G[ES_m])) \to L(S_{f[i]}) = S_{f[i]},$$
$$L:\ (S_{f[i]}, S_{f[j]}) \in E(G[ES_m])) \to S_{f[i]} \bigcap S_{f[j]} \ for\ integers\ 1 \leq i \neq j \leq m.$$

Applying Theorem 3.1, a conclusion following can be readily obtained.

Theorem 3.3 *A system (ES_m) consisting of equations in (ES_{m_i}), $1 \leq i \leq m$ is solvable if and only if $G[ES_m] \simeq K_m$ with $\emptyset \neq S \subset \bigcap\limits_{i=1}^{m} S_{f[i]}$. Otherwise, non-solvable, i.e., $G[ES_m] \not\simeq K_m$, or $G[ES_m] \simeq K_m$ but $\bigcap\limits_{i=1}^{m} S_{f[i]} = \emptyset$.*

Let $T:\ (x_1, x_2, \cdots, x_n) \to (x_1', x_2', \cdots, x_n')$ be linear transformation determined by an invertible matrix $[a_{ij}]_{n\times n}$, i.e., $x_i' = a_{i1}x_1 + a_{i2}x_2 + \cdots + a_{in}x_n$, $1 \leq i \leq n$ and let $T(S_{f[k]}) = S'_{f[k]}$ for integers $1 \leq k \leq m$. Clearly, $T:\ \{S_{f[i]},\ 1 \leq i \leq m\} \to \{S'_{f[i]},\ 1 \leq i \leq m\}$ and

$$S'_{f[i]} \bigcap S'_{f[j]} \neq \emptyset \ \text{ if and only if } \ S_{f[i]} \bigcap S_{f[j]} \neq \emptyset$$

for integers $1 \leq i \neq j \leq m$. Consequently, if $T:\ (ES_m) \leftarrow ('ES_m)$, then $G[ES_m] \simeq G['ES_m]$. Thus T induces an isomorphism T^* of graph from $G[ES_m]$ to $G['ES_m]$, which implies the following result:

Theorem 3.4 *A system (ES_m) of equations $f_i(\overline{x}) = 0, 1 \leq i \leq m$ inherits an invariant $G[ES_m]$ under the action of invertible linear transformations on $\mathbb{R}^n$.*

Theorem 3.4 enables one to introduce a definition following for algebraic system (ES_m) of equations, which expands the scope of algebraic equations.

Definition 3.5 *If $G[ES_m]$ is the topological graph of system (ES_m) consisting of equations in (ES_{m_i}) for integers $1 \leq i \leq m$, introduced in Definition 3.2, then $G[ES_m]$ is called a G-solution of system (ES_m).*

Thus, for developing the theory of algebraic equations, a central problem in front of one should be:

Problem 3.6 *For an equation system (ES_m), determine its G-solution $G[ES_m]$.*

For example, the solvable system (ES_m) in classical algebra is nothing else but a K_m-solution with $\bigcap\limits_{i=1}^{m} S_{f[i]} \neq \emptyset$, as claimed in Theorem 3.3. The readers are refereed to references [22] or [26] for more results on non-solvable equations.

3.2 Homogenous Equations

A system (ES_m) is homogenous if each of its equations $f_i(x_0, x_1, \cdots, x_n),\ 1 \leq i \leq m$ is homogenous, i.e.,

$$f_i(\lambda x_0, \lambda x_1, \cdots, \lambda x_n) = \lambda^d f_i(x_0, x_1, \cdots, x_n)$$

for a constant λ, denoted by (hES_m). For such a system, there are always existing a K_m-solution with $\{x_i = 0,\ 0 \leq i \leq n\} \subset \bigcap\limits_{i=1}^{m} S_{f[i]}$ and each $f_i(x_0, x_1, \cdots, x_n) = 0$ passes through $O = (\underbrace{0, 0, \cdots, 0}_{n+1})$ in $\mathbb{R}^n$. Clearly, an invertible linear transformation T action on such a K_m-solution is also a K_m-solution.

However, there are meaningless for such a K_m-solution in projective space $\mathbb{P}^n$ because $O \notin \mathbb{P}^n$. Thus, new invariants for such systems under projective transformations $(x'_0, x'_1, \cdots, x'_n) = [a_{ij}]_{(n+1)\times(n+1)}(x_0, x_1, \cdots, x_n)$ should be found, where $[a_{ij}]_{(n+1)\times(n+1)}$ is invertible. In $\mathbb{R}^2$, two lines $P(x, y), Q(x, y)$ are *parallel* if they are not intersect. But in $\mathbb{P}^2$, this parallelism will never appears because the Bézout's theorem claims that any two curves $P(x, y, z),\ Q(x, y, z)$ of degrees m, n without common components intersect precisely in mn points. However, denoted by $I(P, Q)$ the set of intersections of homogenous polynomials $P(\overline{x})$ with $Q(\overline{x})$ with $\overline{x} = (x_0, x_1, \cdots, x_n)$. The parallelism in $\mathbb{R}^n$ can be extended to $\mathbb{P}^n$ following, which enables one to find invariants on systems homogenous equations.

Definition 3.7 *Let $P(\overline{x}), Q(\overline{x})$ be two complex homogenous polynomials of degree d with $\overline{x} = (x_0, x_1, \cdots, x_n)$. They are said to be parallel, denoted by $P \parallel Q$ if $d \geq 1$ and there are constants $a, b, \cdots, c$ (not all zero) such that for $\forall \overline{x} \in I(P, Q)$, $ax_0 + bx_1 + \cdots + cx_n = 0$, i.e., all intersections of $P(\overline{x})$ with $Q(\overline{x})$ appear at a hyperplane on $\mathbb{P}^n\mathbf{C}$, or $d = 1$ with all intersections at the infinite $x_n = 0$. Otherwise, $P(\overline{x})$ are not parallel to $Q(\overline{x})$, denoted by $P \nparallel Q$.*

Definition 3.8 *Let $P_1(\overline{x}) = 0, P_2(\overline{x}) = 0, \cdots, P_m(\overline{x}) = 0$ be homogenous equations in (hES_m). Define a topological graph $G[hES_m]$ in $\mathbb{P}^n$ by*

$$V(G[hES_m]) = \{P_1(\overline{x}), P_2(\overline{x}), \cdots, P_m(\overline{x})\};$$
$$E(G[hES_m]) = \{(P_i(\overline{x}), P_j(\overline{x})) | P_i \nparallel P_j,\ 1 \leq i, j \leq m\}$$

with a labeling

$$L:\ P_i(\overline{x}) \to P_i(\overline{x}), \quad (P_i(\overline{x}), P_j(\overline{x})) \to I(P_i, P_j),\ \textit{where}\ 1 \leq i \neq j \leq m.$$

For any system (hES_m) of homogenous equations, $G[hES_m]$ is an indeed invariant under the action of invertible linear transformations T on $\mathbb{P}^n$. By definition in [6], a *covariant* $C(a_{\overline{k}}, \overline{x})$ on homogenous polynomials $P(\overline{x})$ is a polynomial function of coefficients $a_{\overline{k}}$ and variables $\overline{x}$. We furthermore find a topological invariant on covariants following.

Theorem 3.9 *Let (hES_m) be a system consisting of covariants $C_i(a_{\overline{k}}, \overline{x})$ on homogenous polynomials $P_i(\overline{x})$ for integers $1 \leq i \leq m$. Then, the graph $G[hES_m]$ is a covariant under the action of invertible linear transformations T, i.e., for $\forall C_i(a_{\overline{k}}, \overline{x}) \in (ES_m)$, there is $C_{i'}(a_{\overline{k}}, \overline{x}) \in (ES_m)$ with*

$$C_{i'}(a'_{\overline{k}}, \overline{x}') = \Delta^p C_i(a_{\overline{k}}, \overline{x})$$

holds for integers $1 \leq i \leq m$, where p is a constant and Δ is the determinant of T.

Proof Let $G^T[hES_m]$ be the topological graph on transformed system $T(hES_m)$ defined in Definition 3.8. We show that the invertible linear transformation T naturally induces an isomorphism between graphs $G[hES_m]$ and $G^T[hES_m]$. In fact, T naturally induces a mapping $T^*: G[hES_m] \to G^T[hES_m]$ on $\mathbb{P}^n$. Clearly, $T^*: V(G[hES_m]) \to V(G^T[hES_m])$ is $1-1$, also onto by definition. In projective space $\mathbb{P}^n$, a line is transferred to a line by an invertible linear transformation. Therefore, $C_u^T \parallel C_v^T$ in $T(ES_m)$ if and only if $C_u \parallel C_v$ in (hES_m), which implies that $(C_u^T, C_v^T) \in E(G^T[ES_m])$ if and only if $(C_u, C_v) \in E(G[hES_m])$. Thus, $G[hES_m] \simeq G^T[hES_m]$ with an isomorphism T^* of graph.

Notice that $I\left(C_u^T, C_v^T\right) = T\left(I(C_u, C_v)\right)$ for $\forall (C_u, C_v) \in E(G[hES_m])$. Consequently, the induced mapping

$$T^*: V(G[hES_m]) \to V(G^T[hES_m]), \qquad E(G[hES_m]) \to E(G^T[hES_m])$$

is commutative with that of labeling L, i.e., $T^* \circ L = L \circ T^*$. Thus, T^* is an isomorphism from topological graph $G[hES_m]$ to $G^T[hES_m]$. □

Particularly, let $p = 0$, i.e., (ES_m) consisting of homogenous polynomials $P_1(\overline{x}), P_2(\overline{x}), \cdots, P_m(\overline{x})$ in Theorem 3.9. Then we get a result on systems of homogenous equations following.

Corollary 3.10 *A system (hES_m) of homogenous equations $f_i(\overline{x}) = 0, 1 \leq i \leq m$ inherits an invariant $G[hES_m]$ under the action of invertible linear transformations on $\mathbb{P}^n$.*

Thus, for homogenous equation systems (hES_m), the *G-solution* in Problem 3.6 should be substituted by $G[hES_m]$*-solution*.

§4. Differential Equations

4.1 Non-Solvable Ordinary Differential Equations

For integers $m,\ n \geq 1$, let

$$\dot{X} = F_i(X),\ 1 \leq i \leq m \qquad (DES_m^1)$$

be a differential equation system with continuous $F_i: \mathbf{R}^n \to \mathbf{R}^n$, $\dot{X} = \dfrac{dX}{dt}$ such that $F_i(\overline{0}) = \overline{0}$, particularly, let

$$\dot{X} = A_1X, \cdots, \dot{X} = A_kX, \cdots, \dot{X} = A_mX \qquad (LDES_m^1)$$

be a linear ordinary differential equation system of first order with

$$\dot{X} = (\dot{x}_1, \dot{x}_2, \cdots, \dot{x}_n)^t = \left(\frac{dx_1}{dt}, \frac{dx_2}{dt}, \cdots, \frac{dx_n}{dt}\right)$$

and

$$\begin{cases} x^{(n)} + a_{11}^{[0]}x^{(n-1)} + \cdots + a_{1n}^{[0]}x = 0 \\ x^{(n)} + a_{21}^{[0]}x^{(n-1)} + \cdots + a_{2n}^{[0]}x = 0 \\ \cdots\cdots\cdots\cdots \\ x^{(n)} + a_{m1}^{[0]}x^{(n-1)} + \cdots + a_{mn}^{[0]}x = 0 \end{cases} \qquad (LDE_m^n)$$

a linear differential equation system of order n with

$$A_k = \begin{bmatrix} a_{11}^{[k]} & a_{12}^{[k]} & \cdots & a_{1n}^{[k]} \\ a_{21}^{[k]} & a_{22}^{[k]} & \cdots & a_{2n}^{[k]} \\ \cdots & \cdots & \cdots & \cdots \\ a_{n1}^{[k]} & a_{n2}^{[k]} & \cdots & a_{nn}^{[k]} \end{bmatrix} \quad \text{and} \quad X = \begin{bmatrix} x_1(t) \\ x_2(t) \\ \cdots \\ x_n(t) \end{bmatrix},$$

where, $x^{(n)} = \dfrac{d^n x}{dt^n}$, all $a_{ij}^{[k]}$, $0 \le k \le m,\ 1 \le i,j \le n$ are numbers. Such a system (DES_m^1) or $(LDES_m^1)$ (or (LDE_m^n)) are called *non-solvable* if there are no function $X(t)$ (or $x(t)$) hold with (DES_m^1) or $(LDES_m^1)$ (or (LDE_m^n)) unless constants. For example, the following differential equation system

$$(LDE_6^2)\begin{cases} \ddot{x} - 3\dot{x} + 2x = 0 & (1) \\ \ddot{x} - 5\dot{x} + 6x = 0 & (2) \\ \ddot{x} - 7\dot{x} + 12x = 0 & (3) \\ \ddot{x} - 9\dot{x} + 20x = 0 & (4) \\ \ddot{x} - 11\dot{x} + 30x = 0 & (5) \\ \ddot{x} - 7\dot{x} + 6x = 0 & (6) \end{cases}$$

is a non-solvable system.

According to theory of ordinary differential equations ([32]), any linear differential equation system $(LDES_1^1)$ of first order in $(LDES_m^1)$ or any differential equation (LDE_1^n) of order n with complex coefficients in (LDE_m^n) are solvable with a solution basis $\mathscr{B} = \{\ \overline{\beta}_i(t) |\ 1 \le i \le n\}$ such that all general solutions are linear generated by elements in $\mathscr{B}$.

Denoted the solution basis of systems (DES_m^1) or $(LDES_m^1)$ (or (LDE_m^n)) of ordinary differential equations by $\mathscr{B}_1, \mathscr{B}_2, \cdots, \mathscr{B}_m$ and define a topological graph $G[DES_m^1]$ or $G[LDES_m^1]$ (or $G[LDE_m^n]$) in $\mathbb{R}^n$ by

$$\begin{aligned} V(G[DES_m^1]) &= V(G[LDES_m^1]) = V(G[LDE_m^n]) = \{\mathscr{B}_1, \mathscr{B}_2, \cdots, \mathscr{B}_m\}; \\ E(G[DES_m^1]) &= E(G[LDES_m^1]) = E(G[LDE_m^n]) \\ &= \{(\mathscr{B}_i, \mathscr{B}_j) \ \text{ if } \ \mathscr{B}_i \bigcap \mathscr{B}_j \ne \emptyset,\ 1 \le i,j \le m\} \end{aligned}$$

with a labeling

$$L:\ \mathscr{B}_i \to \mathscr{B}_i, \qquad (\mathscr{B}_i, \mathscr{B}_j) \to \mathscr{B}_i \bigcap \mathscr{B}_j \text{ for } 1 \le i \ne j \le m.$$

Let T be a linear transformation on $\mathbb{R}^n$ determined by an invertible matrix $[a_{ij}]_{n\times n}$. Let

$$T:\ \{\mathscr{B}_i,\ 1 \le i \le m\} \to \{\mathscr{B}'_i,\ 1 \le i \le m\}.$$

It is clear that $\mathscr{B}'_i$ is the solution basis of the ith transformed equation in (DES_m^1) or $(LDES_m^1)$ (or (LDE_m^n)), and $\mathscr{B}'_i \bigcap \mathscr{B}'_j \ne \emptyset$ if and only if $\mathscr{B}_i \bigcap \mathscr{B}_j \ne \emptyset$. Thus T naturally induces an isomorphism T^* of graph with $T^* \circ L = L \circ T^*$ on labeling L.

Theorem 4.1 *A system (DES_m^1) or $(LDES_m^1)$ (or (LDE_m^n)) of ordinary differential equations inherits an invariant $G[DES_m^1]$ or $G[LDES_m^1]$ (or $G[LDE_m^n]$) under the action of invertible linear transformations on $\mathbb{R}^n$.*

Clearly, if the topological graph $G[DES_m^1]$ or $G[LDES_m^1]$ (or $G[LDE_m^n]$) are determined, the global behavior of solutions of systems (DES_m^1) or $(LDES_m^1)$ (or (LDE_m^n)) in $\mathbb{R}^n$ are readily known. Such graphs are called respectively $G[DES_m^1]$*-solution* or $G[LDES_m^1]$*-solution* (or $G[LDE_m^n]$*-solution*) of systems of (DES_m^1) or $(LDES_m^1)$ (or (LDE_m^n)). Thus, for developing ordinary differential equation theory, an interesting problem should be:

Problem 4.2 *For a system of (DES_m^1) (or $(LDES_m^1)$, or (LDE_m^n)) of ordinary differential equations, determine its $G[DES_m^1]$-solution (or $G[LDES_m^1]$-solution, or $G[LDE_m^n]$-solution).*

For example, the topological graph $G[LDE_6^2]$ of system (LDE_6^2) of linear differential equation of order 2 in previous is shown in Fig.7.

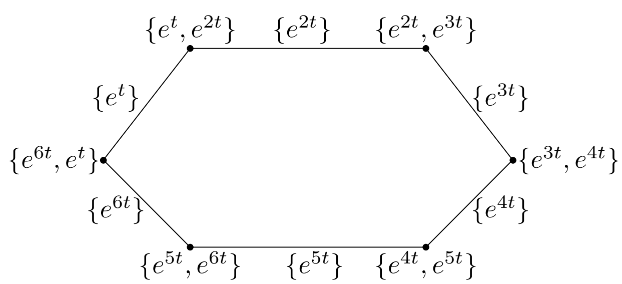


Fig.7

4.2 Non-Solvable Partial Differential Equations

Let $L_1, L_2, \cdots, L_m$ be m partial differential operators of first order (linear or non-linear) with

$$L_k = \sum_{i=1}^{n} a_{ki}\frac{\partial}{\partial x_i},\ 1 \leq k \leq m.$$

Then the system of partial differential equations

$$L_i[u(x_1, x_2, \cdots, x_n)] = h_i, \qquad 1 \leq i \leq m, \qquad (PDES_m)$$

or the Cauchy problem

$$\begin{cases} L_i[u] = h_i \\ u(x_1, x_2, \cdots, x_{n-1}, x_n^0) = \varpi_i, \quad 1 \leq i \leq m \end{cases} \qquad (PDES_m^C)$$

is *non-solvable* if there are no function $u(x_1, \cdots, x_n)$ on a domain $D \subset \mathbb{R}^n$ with $(PDES_m)$ or $(PDES_m^C)$ holds, where $h_i,\ 1 \leq i \leq m$ and $\varpi_i\ 1 \leq i \leq m$ are all continuous functions on $D \subset \mathbb{R}^n$.

Clearly, the ith partial differential equation is solvable [3]. Denoted by S_i^0 the solution of ith equation in $(PDES_m)$ or $(DEPS_m^C)$. Then the system $(PDES_m)$ or $(DEPS_m^C)$ of partial differential equations is solvable only if $\bigcap\limits_{i=1}^{m} S_i^0 \neq \emptyset$. Because $u : \mathbb{R}^n \to \mathbb{R}^n$ is differentiable, so the $(PDES_m)$ or $(DEPS_m^C)$ is solvable only if $\bigcap\limits_{i=1}^{m} S_i^0$ is a non-empty functional set on a domain $D \subset \mathbb{R}^n$. Otherwise, non-solvable, i.e., $\bigcap\limits_{i=1}^{m} S_i^0 = \emptyset$ for any domain $D \subset \mathbb{R}^n$.

Define a topological graph $G[PDES_m]$ or $G[DEPS_m^C]$ in $\mathbb{R}^n$ by

$$\begin{aligned} V(G[PDES_m]) &= V(G[DEPS_m^C]) = \{S_i^0,\ 1 \leq i \leq m\}; \\ E(G[PDES_m]) &= E(G[DEPS_m^C]) \\ &= \{(S_i^0, S_j^0) \text{ if } S_i^0 \bigcap S_j^0 \neq \emptyset,\ 1 \leq i, j \leq m\} \end{aligned}$$

with a labeling

$$L: \ S_i^0 \to S_i^0, \quad (S_i^0, S_j^0) \in E(G[PDES_m]) = E(G[DEPS_m^C]) \to S_i^0 \bigcap S_j^0$$

for $1 \leq i \neq j \leq m$. Similarly, if T is an invertible linear transformation on $\mathbb{R}^n$, then $T(S_i^0)$ is the solution of ith transformed equation in $(PDES_m)$ or $(DEPS_m^C)$, and $T(S_i^0) \bigcap T(S_j^0) \neq \emptyset$ if and only if $S_i^0 \bigcap S_j^0 \neq \emptyset$. Accordingly, T induces an isomorphism T^* of graph with $T^* \circ L = L \circ T^*$ holds on labeling L. We get the following result.

Theorem 4.3 *A system $(PDES_m)$ or $(DEPS_m^C)$ of partial differential equations of first order inherits an invariant $G[PDES_m]$ or $G[DEPS_m^C]$ under the action of invertible linear transformations on $\mathbb{R}^n$.*

Such a topological graph $G[PDES_m]$ or $G[DEPS_m^C]$ are said to be the $G[PDES_m]$-*solution* or $G[DEPS_m^C]$-*solution* of systems $(PDES_m)$ and $(DEPS_m^C)$, respectively. For example, the $G[DEPS_3^C]$-solution of Cauchy problem

$$\begin{cases} u_t + au_x = 0 \\ u_t + xu_x = 0 \\ u_t + au_x + e^t = 0 \\ u|_{t=0} = \phi(x) \end{cases} \qquad (DEPS_3^C)$$

is shown in Fig.8

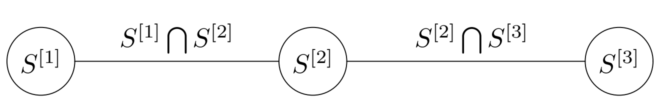


Fig.8

Clearly, system $(DEPS_3^C)$ is contradictory because $e^t \neq 0$ for t. However,

$$\begin{cases} u_t + au_x = 0 \\ u|_{t=0} = \phi(x) \end{cases} \quad \begin{cases} u_t + xu_x = 0 \\ u|_{t=0} = \phi(x) \end{cases} \quad \text{and} \quad \begin{cases} u_t + au_x + e^t = 0 \\ u|_{t=0} = \phi(x) \end{cases}$$

are solvable with respective solutions $S^{[1]}=\{\phi(x-at)\}$, $S^{[2]}=\{\phi(\frac{x}{e^t})\}$ and $S^{[3]}=\{\phi(x-at)-e^t+1\}$, and $S^{[1]}\bigcap S^{[2]}=\{\phi(x-at)=\phi(\frac{x}{e^t})\}$, $S^{[2]}\bigcap S^{[3]}=\{\phi(\frac{x}{e^t})=\phi(x-at)-e^t+1\}$, but $S^{[1]}\bigcap S^{[3]}=\emptyset$.

Similar to ordinary case, an interesting problem on partial differential equations is the following:

Problem 4.4 *For a system of ($PDES_m$) or ($DEPS_m^C$) of partial differential equations, determine its $G[PDES_m]$-solution or $G[DEPS_m^C]$-solution.*

It should be noted that for an algebraically contradictory linear system

$$\begin{cases} F_i(x_1,\cdots,x_n,u,p_1,\cdots,p_n,)=0\\ F_j(x_1,\cdots,x_n,u,p_1,\cdots,p_n,)=0,\end{cases}$$

if

$$F_k(x_1,\cdots,x_n,u,p_1,\cdots,p_n,)=0$$

is contradictory to one of there two partial differential equations, then it must be contradictory to another. This fact enables one to classify equations in ($LPDES_m$) by the contradictory property and determine $G[LPDES_m^C]$. Thus if $\mathscr{C}_1,\cdots,\mathscr{C}_l$ are maximal contradictory classes for equations in ($LPDES$), then $G[LPDES_m^C]\simeq K(\mathscr{C}_1,\cdots,\mathscr{C}_l)$, i.e., an l-partite complete graph. Accordingly, all $G[LPDES_m^C]$-solutions of linear systems ($LPDES_m$) are nothing else but $K(\mathscr{C}_1,\cdots,\mathscr{C}_s)$-solutions. More behaviors on non-solvable ordinary or partial differential equations of first order, for instance the global stability can be found in references [25]-[27].

4.3 Equation's Combinatorics

All these discussions in Sections 3 and $4.2-4.3$ lead to a conclusion that *a non-solvable system (ES) of equations in n variables inherits an invariant $G[ES]$ of topological graph labeled with those of individually solutions, if it is individually solvable*, i.e., equation's combinatorics by view it with the topological graph $G[ES]$ in $\mathbb{R}^n$. Thus, for holding the global behavior of a system (ES) of equations, the right way is not just to determine it is solvable or not, but its $G[ES]$-solution. Such a $G[ES]$-solution is existent by philosophy and enables one to include non-solvable equations, no matter what they are algebraic, differential, integral or operator equations to mathematics by G-system following:

Definition 4.5 *A G-system (ES_m) of equations $O_i(\overline{X})=\overline{0}$, $1\le i\le m$ with constraints $\mathcal{C}$ is a topological graph G with labeling $L:\ v\in V(G)\to L(v)\in\{S_{O_i};\ 1\le i\le m\}$ and $L:\ (u,v)\in E(G)\to L(u)\bigcap L(v)$ with $L(u)\bigcap L(v)\neq\emptyset$, denoted by $G[ES_m]$, where, S_{O_i} is the solution space of equation $O_i(\overline{X})=\overline{0}$ with constraints $\mathcal{C}$ for integers $1\le i\le m$.*

Thus, holding the true face of a thing T characterized by a system (ES_m) of equations needs one to determine its G-system, i.e., $G[ES_m]$-solution, not only solvable or not for its objective reality.

Problem 4.6 *Determine $G[ES_m]$ for equation systems (ES_m), such as those of algebraic,* *differential, integral, operator equations, or their combination, or conversely, characterize G-systems of equations for given graphs G, foe example, these G-systems of equations for complete graphs $G = K_m$, complete bipartite graph $K(n_1, n_2)$ with $n_1 + n_2 = m$, path P_{m-1} or circuit C_m.*

By this view, a solvable system (ES_m) of equations in classical mathematics is nothing else but such a K_m-system with $\bigcap\limits_{e\in E(K_m)} L(e) \neq \emptyset$. However, as we known, more systems of equations established on characters $\mu_i,\ 1 \leq i \leq n$ for a thing T are non-solvable with contradictions if $n \geq 2$. It is nearly impossible to solve all those systems in classical mathematics. Even so, its G-systems reveals behaviors of thing T to human beings.

§5. Geometry

As what one sees with an immediately form on things, the geometry proves to be one of applicable means for portraying things by its homogeneity with distinction. Nevertheless, the non-geometry can also contributes describing things complying with the Erlangen Programme that of Klein.

5.1 Non-Spaces

Let $\mathscr{K}^n = \{(x_1, x_2, \cdots, x_n)\}$ be an n-dimensional Euclidean (affine or projective) space with a normal basis $\overline{\epsilon}_i,\ 1 \leq i \leq n$, $\overline{x} \in \mathscr{K}^n$ and let $\overrightarrow{V}_{\overline{x}}$, ${}_{\overline{x}}\overrightarrow{V}$ be two orientation vectors with end or initial point at $\overline{x}$. Such as those shown in Fig.9.

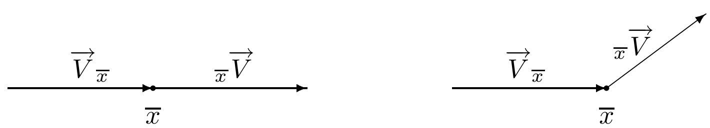


Fig.9

For point $\forall \overline{x} \in \mathscr{K}^n$, we associate it with an invertible linear mapping

$$\mu : \{\overline{\epsilon}_1, \overline{\epsilon}_2, \cdots, \overline{\epsilon}_n\} \to \{\overline{\epsilon}'_1, \overline{\epsilon}'_2, \cdots, \overline{\epsilon}'_n\}$$

such that $\mu(\overline{\epsilon}_i) = \overline{\epsilon}'_i,\ 1 \leq i \leq n$, called its *weight*, i.e.,

$$(\overline{\epsilon}'_1, \overline{\epsilon}'_2, \cdots, \overline{\epsilon}'_n) = [a_{ij}]_{n\times n} (\overline{\epsilon}_1, \overline{\epsilon}_2, \cdots, \overline{\epsilon}_n)^t$$

where, $[a_{ij}]_{n\times n}$ is an invertible matrix. Such a space is a weighted space on points in $\mathscr{K}^n$, denoted by $(\mathscr{K}^n, \mu)$ with $\mu :\ \overline{x} \to \mu(\overline{x}) = [a_{ij}]_{n\times n}$. Clearly, if $\mu(\overline{x}_1) = \left[a'_{ij}\right]$, $\mu(\overline{x}_2) = \left[a''_{ij}\right]$, then $\mu(\overline{x}_1) = \mu(\overline{x}_2)$ if and only if there exists a constant λ such that $\left[a'_{ij}\right]_{n\times n} = \left[\lambda a''_{ij}\right]_{n\times n}$, and $(\mathscr{K}^n, \mu) = \mathbb{R}^n$ ($\mathbb{A}^n$ or $\mathbb{P}^n$), i.e., n-dimensional *Euclidean* (*affine* or *projective space*) if and only if $[a_{ij}]_{n\times n} = I_{n\times n}$ for $\forall \overline{x} \in \mathscr{K}^n$. Otherwise, *non-Euclidean, non-affine or non-projective space*, abbreviated to *non-space*.

Notice that $\left[a'_{ij}\right]_{n\times n} = \left[\lambda a''_{ij}\right]_{n\times n}$ is an equivalent relation on invertible $n \times n$ matrixes. Thus, for $\forall \overline{x}_0 \in \mathscr{K}^n$, define

$$\mathscr{C}(\overline{x}_0) = \{\overline{x} \in \mathscr{K}^n | \mu(\overline{x}) = \lambda\mu(\overline{x}_0),\ \lambda \in \mathbb{R}\},$$

an *equivalent set* of points to $\overline{x}_0$. Then there exist representatives $\mathscr{C}_\kappa, \kappa \in \Lambda$ constituting a partition of $\mathscr{K}^n$ in all equivalent sets $\mathscr{C}(\overline{x}), \overline{x} \in \mathscr{K}^n$ of points, i.e.,

$$\mathscr{K}^n = \bigcup_{\kappa\in\Lambda} \mathscr{C}_\kappa \quad \text{with} \quad \mathscr{C}_{\kappa_1} \bigcap \mathscr{C}_{\kappa_2} = \emptyset \ \text{ for } \ \kappa_1, \kappa_2 \in \Lambda \ \text{ if } \ \kappa_1 \neq \kappa_2,$$

where Λ maybe countable or uncountable.

Let $\mu(\overline{x}) = [a_{ij}]_{n\times n} = A_\kappa$ for $\overline{x} \in \mathscr{C}_\kappa$. For viewing behaviors of orientation vectors in an equivalent set $\mathscr{C}_\kappa$ of points, define $\mu_{A_\kappa}: \ \mathscr{K}^n \to \mu_{A_\kappa}(\mathscr{K}^n)$ by $\mu_{A_\kappa}(\overline{x}) = A_\kappa$. Then $(\mathscr{K}^n, \mu_{A_\kappa})$ is also a non-space if $A_\kappa \neq I_{n\times n}$. However, $(\mathscr{K}^n, \mu_{A_\kappa})$ approximates to $\mathscr{K}^n$ with homogeneity because each orientation vector only turns a same direction passing through a point. Thus, $(\mathscr{K}^n, \mu_{A_\kappa})$ can be viewed as space $\mathscr{K}^n$, denoted by $\mathscr{K}^n_{\mu_A}$. Define a topological graph $G[\mathscr{K}^n, \mu]$ by

$$V(G[\mathscr{K}^n, \mu]) = \{\mathscr{K}^n_{\mu_\kappa}, \ \ \kappa \in \Lambda\};$$
$$E(G[\mathscr{K}^n, \mu]) = \{(\mathscr{K}^n_{\mu_{\kappa_1}}, \mathscr{K}^n_{\mu_{\kappa_2}}) \text{ if } \mathscr{K}^n_{\mu_{\kappa_1}} \bigcap \mathscr{K}^n_{\mu_{\kappa_2}} \neq \emptyset,\ \kappa_1, \kappa_2 \in \Lambda, \kappa_1 \neq \kappa_2\}$$

with labels

$$L: \ \mathscr{K}^n_{\mu_\kappa} \in V(G[\mathscr{K}^n, \mu]) \to \mathscr{K}^n_{\mu_\kappa},$$
$$L: \ (\mathscr{K}^n_{\mu_{\kappa_1}}, \mathscr{K}^n_{\mu_{\kappa_2}}) \in E(G[\mathscr{K}^n, \mu]) \to \mathscr{K}^n_{\mu_{\kappa_1}} \bigcap \mathscr{K}^n_{\mu_{\kappa_2}}, \ \ \kappa_1 \neq \kappa_2 \in \Lambda.$$

Then, we get an overview on $(\mathscr{K}^n, \mu)$ with Euclidean spaces $\mathscr{K}^n_{\mu_\kappa}, \kappa \in \Lambda$ by combinatorics. Clearly, $\mathscr{K}^n \bigcap \mathscr{K}^n_{\mu_\kappa} = \mathscr{C}_\kappa$ and $\mathscr{K}^n_{\mu_{\kappa_1}} \bigcap \mathscr{K}^n_{\mu_{\kappa_2}} = \emptyset$ if none of $\mathscr{K}^n_{\mu_{\kappa_1}}, \mathscr{K}^n_{\mu_{\kappa_2}}$ being $\mathscr{K}^n$. Thus, $G[\mathscr{K}^n, \mu] \simeq K_{1,|\Lambda|-1}$, a star with center $\mathscr{K}^n$, such as those shown in Fig.10. Otherwise, $G[\mathscr{K}^n, \mu] \simeq \overline{K}_{|\Lambda|}$, i.e., $|\Lambda|$ isolated vertices, which can be turned into $K_{1,|\Lambda|}$ by adding an imaginary center vertex $\mathscr{K}^n$.

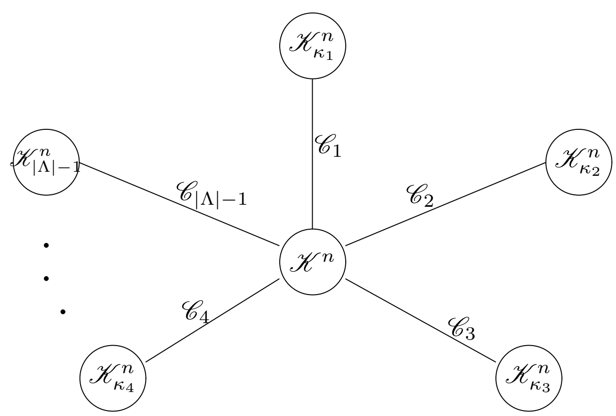


Fig.10

Let T be an invertible linear transformation on $\mathscr{K}^n$ determined by $(\overline{x}') = [\alpha_{ij}]_{n\times n}(\overline{x})^t$. Clearly, $T: \mathscr{K}^n \to \mathscr{K}^n$, $\mathscr{K}^n_{\mu_\kappa} \to T(\mathscr{K}^n_\kappa)$ and $T(\mathscr{K}^n_{\kappa_1})\bigcap T(\mathscr{K}^n_{\kappa_2}) \neq \emptyset$ if and only if $\mathscr{K}^n_{\kappa_1}\bigcap \mathscr{K}^n_{\kappa_2} \neq \emptyset$. Furthermore, one of $T(\mathscr{K}^n_{\kappa_1})$, $T(\mathscr{K}^n_{\kappa_2})$ should be $\mathscr{K}^n$. Thus T induces an isomorphism T^* from $G[\mathscr{K}^n,\mu]$ to $G[T(\mathscr{K}^n),\mu]$ of graph. Accordingly, we know the result following.

Theorem 5.1 *An n-dimensional non-space $(\mathscr{K}^n,\mu)$ inherits an invariant $G[\mathscr{K}^n,\mu]$, i.e., a star $K_{1,|\Lambda|-1}$ or $K_{1,|\Lambda|}$ under the action of invertible linear transformations on $\mathbb{K}^n$, where Λ is an index set such that all equivalent sets $\mathscr{C}_\kappa, \kappa\in\Lambda$ constitute a partition of space $\mathscr{K}^n$.*

5.2 Non-Manifolds

Let M be an n-dimensional manifold with an alta $\mathscr{A} = \{\ (U_\lambda;\varphi_\lambda) \mid \lambda\in\Lambda\}$, where $\varphi_\lambda:\ U_\lambda \to \mathbb{R}^n$ is a homeomorphism with countable Λ. A *non-manifold* $\neg M$ on M is such a topological space with $\varphi:\ U_\lambda \to \mathbb{R}^{n_\lambda}$ for integers $n_\lambda \geq 1,\ \lambda\in\Lambda$, which is a special but more applicable case of non-space $(\mathbb{R}^n,\mu)$. Clearly, if $n_\lambda = n$ for $\lambda\in\Lambda$, $\neg M$ is nothing else but an n-manifold.

For an n-manifold M, each U_λ is itself an n-manifold for $\lambda\in\Lambda$ by definition. Generally, let M_λ be an n_λ-manifold with an alta $\mathscr{A}_\lambda = \{\ (U_{\lambda\kappa};\varphi_{\lambda\kappa}) \mid \kappa\in\Lambda_\lambda\}$, where $\varphi_{\lambda\kappa}:\ U_{\lambda\kappa} \to \mathbb{R}^{n_\lambda}$. A *combinatorial manifold* $\widetilde{M}$ on M is such a topological space constituted by $M_\lambda,\ \lambda\in\Lambda$. Clearly, $\bigcup\limits_{\lambda\in\Lambda}\Lambda_\lambda$ is countable. If $n_\lambda = n$, i.e., all M_λ is an n-manifold for $\lambda\in\Lambda$, then the union $\mathscr{M}$ of $M_\lambda,\ \lambda\in\Lambda$ is also an n-manifold with alta

$$\widetilde{\mathscr{A}} = \bigcup_{\lambda\in\Lambda}\mathscr{A}_\lambda = \{(U_{\lambda\kappa};\varphi_{\lambda\kappa}) \mid \kappa\in\Lambda_\lambda,\ \lambda\in\Lambda\}.$$

Theorem 5.2 *A combinatorial manifold $\widetilde{M}$ is a non-manifold on $\mathscr{M}$, i.e.,*

$$\widetilde{M} = \neg\mathscr{M}.$$

Accordingly, we only discuss non-manifolds $\neg M$. Define a topological graph $G[\neg M]$ by

$$V(G[\neg M]) = \{U_\lambda,\quad \lambda\in\Lambda\};$$
$$E(G[\neg M]) = \{(U_{\lambda_1},U_{\lambda_2}) \text{ if } U_{\lambda_1}\bigcap U_{\lambda_2}\neq\emptyset,\ \lambda_1,\lambda_2\in\Lambda, \lambda_1\neq\lambda_2\}$$

with labels

$$L:\ U_\lambda \in V(G[\neg M]) \to U_\lambda,$$
$$L:\ (U_{\lambda_1},U_{\lambda_2}) \in E(\neg M]) \to U_{\lambda_1}\bigcap U_{\lambda_2},\quad \lambda_1\neq\lambda_2\in\Lambda,$$

which is an invariant dependent only on alta $\mathscr{A}$ of M.

Particularly, if each U_λ is a Euclidean spaces $\mathbb{R}^\lambda,\ \lambda\in\Lambda$, we get another topological graph $G[\mathbb{R}^\lambda,\lambda\in\Lambda]$ on Euclidean spaces $\mathbb{R}^\lambda,\ \lambda\in\Lambda$, a special non-manifold called *combinatorial Euclidean space*. The following result on $\neg M$ is easily obtained likewise the proof of Theorem 2.1 in [23].

Theorem 5.3 *A non-manifold $\neg M$ on manifold M with alta $\mathscr{A}=\{\ (U_\lambda;\varphi_\lambda)\mid\lambda\in\Lambda\}$ inherits an topological invariant $G[\neg M]$. Furthermore, if M is locally compact, $G[\neg M]$ is topological homeomorphic to $G[\mathbb{R}^\lambda,\lambda\in\Lambda]$ if $\varphi:\ U_\lambda\to\mathbb{R}^{n_\lambda},\ \lambda\in\Lambda$.*

It should be noted that Whitney proved that an n-manifold can be topological embedded as a closed submanifold of $\mathbb{R}^{2n+1}$ with a sharply minimum dimension $2n+1$ in 1936. Applying this result, one can easily show that a non-manifold $\neg M$ can be embedded into $\mathbb{R}^{2n_{\max}+1}$ if $n_{\max}=\max\{n_\lambda\in\Lambda\}<\infty$. Furthermore, let U_λ itself be a subset of Euclidean space $\mathbb{R}^{n_{\max}+1}$ for $\lambda\in\Lambda$, then $x_{n_{\max}+1}=\varphi_\lambda(x_1,x_2,\cdots,x_{n_\lambda})$ in $\mathbb{R}^{n_{\max}+1}$. Thus, one gets an equation

$$x_{n_{\max}+1}-\varphi_\lambda(x_1,x_2,\cdots,x_{n_\lambda})=0$$

in $\mathbb{R}^{n_{\max}+1}$. Particularly, if $\Lambda=\{1,2,\cdots,m\}$ is finite, one gets a system (ES_m) of equations

$$\begin{cases} x_{n_{\max}+1}-\varphi_\lambda(x_1,x_2,\cdots,x_{n_1})=0\\ x_{n_{\max}+1}-\varphi_\lambda(x_1,x_2,\cdots,x_{n_2})=0\\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\\ x_{n_{\max}+1}-\varphi_\lambda(x_1,x_2,\cdots,x_{n_m})=0\end{cases}\qquad (ES_m)$$

in $\mathbb{R}^{n_{\max}+1}$. Generally, this system (ES_m) is non-solvable, which enables one getting Theorem 3.1 once again.

5.3 Differentiable Non-Manifolds

For $\forall M_\lambda\in\neg M$, if M_λ is differentiable determined by a system of differential equations

$$(DES_{m_\lambda})\begin{cases} F_{\lambda 1}(x_1,x_2,\cdots,x_n,u,u_{x_1},\cdots,u_{x_n},u_{x_1x_2},\cdots,u_{x_1x_n},\cdots)=0\\ F_{\lambda 2}(x_1,x_2,\cdots,x_n,u,u_{x_1},\cdots,u_{x_n},u_{x_1x_2},\cdots,u_{x_1x_n},\cdots)=0\\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\\ F_{\lambda m_\lambda}(x_1,x_2,\cdots,x_n,u,u_{x_1},\cdots,u_{x_n},u_{x_1x_2},\cdots,u_{x_1x_n},\cdots)=0\end{cases}$$

Then the system (DES_m) consisting of systems $(DES_{m_\lambda}),\ 1\leq\lambda\leq m$ of differential equations with prescribed initial values x_{i_0},u_0,p_{i_0} for integers $i=1,2,\cdots,n$ is generally non-solvable with a geometrical figure of differentiable non-manifold $\neg M$.

Notice that a main characters for points p in non-manifold $\neg M$ is that the number of variables for determining its position in space is not a constant. However, it can also introduces differentials on non-manifolds constrained with $\varphi_\kappa|_{U_\kappa\bigcap U_\lambda}=\varphi_\lambda|_{U_\kappa\bigcap U_\lambda}$ for $\forall(U_\kappa,\varphi_\kappa),\ (U_\lambda,\varphi_\lambda)\in\mathscr{A}$, and smooth functions $f:\ \neg M\to\mathbb{R}$ at a point $p\in\neg M$. Denoted respectively by $\mathscr{X}_p$, $T_p\neg M$ all smooth functions and all tangent vectors $\overline{v}:\ \mathscr{X}_p\to\mathbb{R}$ at a point $p\in\neg M$. If $\varphi(p)\in\bigcap\limits_{i=1}^{s}\mathbb{R}^{n_i(p)}$ and $\widehat{s}(p)=\dim(\bigcap\limits_{i=1}^{s}\mathbb{R}^{n_i(p)})$, a simple calculation shows the dimension of tangent vector space

$$\dim T_p\neg M=\widehat{s}(p)+\sum_{i=1}^{s(p)}(n_i-\widehat{s}(p))$$

with a basis

$$\left\{ \left.\frac{\partial}{\partial x_{ij}}\right|_p, \ 1 \leq i \leq s(p), 1 \leq j \leq n_i \text{ with } x_{il} = x_{jl} \text{ if } 1 \leq l \leq \widehat{s}(p) \right\}$$

and similarly, for cotangent vector space $\dim T_p^*\neg M = \dim T_p\neg M$ with a basis

$$\left\{ dx_{ij}|_p, \ 1 \leq i \leq s(p), 1 \leq j \leq n_i \text{ with } x_{il} = x_{jl} \text{ if } 1 \leq l \leq \widehat{s}(p) \right\},$$

which enables one to introduce vector field $\mathscr{X}(\neg M) = \bigcup\limits_{p\in\neg M} \mathscr{X}_p$, tensor field $T_s^r(\neg M) = \bigcup\limits_{p\in\neg M} T_s^r(p,\neg M)$, where,

$$T_s^r(p,\neg M) = \underbrace{T_p\neg M \otimes \cdots \otimes T_p\neg M}_{r} \otimes \underbrace{T_p^*\neg M \otimes \cdots \otimes T_p^*\neg M}_{s}$$

and connection $D: \mathscr{X}(\neg M) \times T_s^r(\neg M) \to T_s^r(\neg M)$ with $D_X\tau = D(X,\tau)$ such that for $\forall X, Y \in \mathscr{X}(\neg M)$, $\tau, \pi \in T_s^r(\neg M)$, $\lambda \in \mathbb{R}$, $f \in C^\infty(\neg M)$,

(1) $D_{X+fY}\tau = D_X\tau + fD_Y\tau$ and $D_X(\tau + \lambda\pi) = D_X\tau + \lambda D_X\pi$;

(2) $D_X(\tau \otimes \pi) = D_X\tau \otimes \pi + \sigma \otimes D_X\pi$;

(3) For any contraction C on $T_s^r(\neg M)$, $D_X(C(\tau)) = C(D_X\tau)$.

Particularly, let $g \in T_2^0(\neg M)$. If g is symmetrical and positive, then $\neg M$ is called a *Riemannian non-manifold*, denoted by $(\neg M, g)$. It can be readily shown that there is a unique connection D on Riemannian non-manifold $(\neg M, g)$ with equality

$$Z(g(X,Y)) = g(D_Z, Y) + g(X, D_Z Y)$$

holds. Such a D with $(\neg M, g)$, denoted by $(\neg M, g, D)$ is called a *Riemannian non-geometry.* Now let $D_{\frac{\partial}{\partial x_{kl}}}\frac{\partial}{\partial x_{ij}} = \Gamma_{(st)}^{(ij)(kl)}\frac{\partial}{\partial x_{ij}}$ on $(U_p;\varphi)$ for point $p \in (\neg M, g, D)$. Then $\Gamma_{(st)}^{(ij)(kl)} = \Gamma_{(st)}^{(kl)(ij)}$ and

$$\Gamma_{st}^{(kl)(ij)} = \frac{1}{2}g_{(st)(uv)}(\frac{\partial g^{(kl)(uv)}}{\partial x_{ij}} + \frac{\partial g^{(uv)(ij)}}{\partial x_{kl}} - \frac{\partial g^{(kl)(ij)}}{\partial x_{uv}}),$$

where $g = g^{(kl)(ij)}dx_{kl}dx_{ij}$ and $g_{(st)(uv)}$ is an element in matrix $[g^{(kl)(ij)}]^{-1}$.

Similarly, a *Riemannian curvature tensor*

$$R: \ \mathscr{X}(\neg M) \times \mathscr{X}(\neg M) \times \mathscr{X}(\neg M) \times \mathscr{X}(\neg M) \to C^\infty(\neg M)$$

of type $(0,4)$ is defined by $R(X,Y,Z,W) = g(R(Z,W)X,Y)$ for $\forall X,Y,Z,W \in \mathscr{X}(\neg M)$ and with a local form

$$R = R^{(ij)(kl)(st)(uv)}dx_{ij} \otimes dx_{kl} \otimes dx_{st} \otimes dx_{uv},$$

where

$$\begin{aligned} R^{(ij)(kl)(st)(uv)} &= \frac{1}{2}\left(\frac{\partial^2 g^{(st)(ij)}}{\partial x_{uv}\partial x_{kl}} + \frac{\partial^2 g^{(uv)(kl)}}{\partial x_{st}\partial x_{ij}} - \frac{\partial^2 g^{(st)(kl)}}{\partial x_{uv}\partial x_{ij}} - \frac{\partial^2 g^{(uv)(ij)}}{\partial x_{st}\partial x_{kl}}\right) \\ &+\Gamma_{ab}^{(st)(ij)}\Gamma_{cd}^{(uv)(kl)}g^{(cd)(ab)} - \Gamma_{ab}^{(st)(kl)}\Gamma^{(uv)(ij)}{}_{cd}\,g^{(cd)(ab)}, \end{aligned}$$

for $\forall p \in \neg M$ and $g^{(ij)(kl)} = g(\frac{\partial}{\partial x_{ij}}, \frac{\partial}{\partial x_{kl}})$, which can be also used for measuring the curved degree of $(\neg M, g, D)$ at point $p \in \neg M$ (see [16] or [21] for details).

Theorem 5.4 *A Riemannian non-geometry $(\neg M, g, D)$ inherits an invariant, i.e., the curvature tensor $R: \mathscr{X}(\neg M) \times \mathscr{X}(\neg M) \times \mathscr{X}(\neg M) \times \mathscr{X}(\neg M) \to C^\infty(\neg M)$.*

5.4 Smarandache Geometry

A fundamental image of geometry $\mathscr{G}$ is that of *space* consisting of point p, line L, plane P, etc. elements with inclusions $P, L \ni p$ and $P \supset L$ and a geometrical axiom is a premise logic function T on geometrical elements $p, L, P, \cdots \in \mathscr{G}$ with $T(p, L, P, \cdots) = 1$ in classical geometry. Contrast to the classic, a *Smarandache geometry* $S\mathscr{G}$ is such a geometry with at least one axiom behaves in two different ways within the same space, i.e., validated and invalided, or only invalided but in multiple distinct ways. Thus, $T(p, L, P, \cdots) = 1$, $\neg T(p, L, P, \cdots) = 1$ hold simultaneously, or $0 < \neg T(p, L, P, \cdots) = I_1, I_2, \cdots, I_k < 1$ for an integer $k \geq 2$ in $S\mathscr{G}$, which enables one to discuss Smarandache geometriy in two cases following:

Case 1. $T(p, L, P, \cdots) = 1 \wedge \neg T(p, L, P, \cdots) = 1$ in $S\mathscr{G}$.

Denoted by $U = T^{-1}(1) \subset S\mathscr{G}$, $V = \neg T^{-1}(1) \subset S\mathscr{G}$. Clearly, if $U \bigcap V \neq \emptyset$ and there are $p, L, P, \cdots \in U \bigcap V$. Then there must be $T(p, L, P, \cdots) = 1$ and $\neg T(p, L, P, \cdots) = 1$ in $U \bigcap V$, a contradiction. Thus, $U \bigcap V = \emptyset$ or $U \bigcap V \neq \emptyset$ but some of elements $p, L, P, \cdots \in S\mathscr{G}$ for T are missed in $U \bigcap V$.

Not loss of generality, let

$$U = \bigoplus_{k=1}^{m} U_C^k \quad \text{and} \quad V = \bigoplus_{i=1}^{n} V_C^i,$$

where U_C^k, V_C^i are respectively connected components in U and V. Define a topological graph $G[U, V]$ following:

$$V(G[U, V]) = \{U_C^k; 1 \leq k \leq m\} \bigcup \{V_C^i; 1 \leq i \leq n\};$$
$$E(G[U, V]) = \{(U_C^k, V_C^i) \text{ if } U_C^k \bigcap V_C^i \neq \emptyset,\ 1 \leq k \leq m, 1 \leq i \leq n\}$$

with labels

$$L: \ U_C^k \in V(G[U, V]) \to U_C^k, \quad V_C^i \in V(G[U, V]) \to V_C^i$$
$$L: \ (U_C^k, V_C^i) \in E(G[U, V]) \to U_C^k \bigcap V_C^i, \quad 1 \leq k \leq m, 1 \leq i \leq n.$$

Clearly, such a graph $G[U, V]$ is bipartite, i.e., $G[U, V] \leq K_{m,n}$ with labels.

Case 2. $0 < \neg T(p, L, P, \cdots) = I_1, I_2, \cdots, I_k < 1$, $k \geq 2$ in $S\mathscr{G}$.

Denoted by $A_1 = \neg T^{-1}(I_1) \subset S\mathscr{G}$, $A_2 = \neg T^{-1}(I_2) \subset S\mathscr{G}$, $\cdots$, $A_k = \neg T^{-1}(I_k) \subset S\mathscr{G}$. Similarly, if $A_i \bigcap A_j \neq \emptyset$ and there are $p, L, P, \cdots \in A_i \bigcap A_j$. Then there must be $A_i \bigcap A_j = \emptyset$ or $A_i \bigcap A_j \neq \emptyset$ but some of elements $p, L, P, \cdots \in S\mathscr{G}$ for T are missed in $A_1 \bigcap A_j$ for integers $1 \leq i \neq j \leq k$.

Let $A_i = \bigoplus_{l=1}^{m_i} A_C^{i_l}$ with $A_C^{i_l},\ 1 \leq l \leq m_i$ connected components in A_i. Define a topological graph $G[A_i,[1,k]]$ following:

$$V(G[A_i,[1,k]]) = \bigcup_{i=1}^{k}\{A_C^{i_l}; 1 \leq l \leq m_i\};$$

$$E(G[A_i,[1,k]]) = \bigcup_{\substack{i,j=1\\ i\neq j}}^{k}\{(A_C^{i_l}, A_C^{j_s}) \text{ if } A_C^{i_l}\bigcap A_C^{j_s} \neq \emptyset,\ 1 \leq l \leq m_i, 1 \leq s \leq m_j\}$$

with labels

$$L:\ A_C^{i_l} \in V(G[A_i,[1,k]]) \to A_C^{i_l},\quad A_C^{j_s} \in V(G[A_i,[1,k]]) \to A_C^{j_s}$$
$$L:\ (A_C^{i_l}, A_C^{j_s}) \in E(G[A_i,[1,k]]) \to A_C^{i_l}\bigcap A_C^{j_s},\quad 1 \leq l \leq m_i, 1 \leq s \leq m_j$$

for integers $1 \leq i \neq j \leq k$. Clearly, such a graph $G[A_i,[1,k]]$ is k-partite, i.e., $G[A_i,[1,k]] \leq K_{m_1,m_2,\cdots,m_k}$ with labels.

For an invertible transformation T on geometry $S\mathscr{G}$, it is clear that $T(p)$, $T(L)$, $T(P), \cdots$ also constitute the elements of $S\mathscr{G}$ with graphs $G[U,V]$ and $G[A_i,[1,k]]$ invariant. Thus, we know

Theorem 5.5 *A Smarandache geometry $S\mathscr{G}$ inherits a bipartite invariant $G[U,V]$ or k-partite $G[A_i,[1,k]]$ under the action of its linear invertible transformations.*

5.5 Geometrical Combinatorics

All previous discussions on non-space $(\mathscr{K}^n, \mu)$, non-manifold $\neg M$ or differentiable non-manifold $\neg M$ and Smarandache geometry $S\mathscr{G}$ allude a philosophical notion that *any non-geometry can be decomposed into geometries inheriting an invariant $G[\mathscr{K}^n, \mu]$, $G[\neg M]$, $G[U,V]$ or $G[A_i,[1,k]]$ of topological graph labeled with those of geometries*, i.e., geometrical combinatorics accordant with that notion of Klein's. Accordingly, for extending field of geometry, one needs to determine the inherited invariants $G[\mathscr{K}^n, \mu]$, $G[\neg M]$, $G[U,V]$ or $G[A_i,[1,k]]$ and then know geometrical behaviors on non-geometries. But this approach is passive for including non-geometry to geometry. A more initiative way with realization is geometrical G-systems following:

Definition 5.6 *Let $(\mathscr{G}_1;\mathcal{A}_1),(\mathscr{G}_2;\mathcal{A}_2,\cdots,(\mathscr{G}_m;\mathcal{A}_m)$ be m geometrical systems, where $\mathscr{G}_i$, $\mathcal{A}_i$ be respectively the geometrical space and the system of axioms for an integer $1 \leq i \leq m$. A geometrical G-system is a topological graph G with labeling $L:\ v \in V(G) \to L(v) \in \{\mathscr{G}_1,\mathscr{G}_2,\cdots,\mathscr{G}_m\}$ and $L:\ (u,v) \in E(G) \to L(u)\bigcap L(v)$ with $L(u)\bigcap L(v) \neq \emptyset$, denoted by $G[\mathscr{G},\mathcal{A}]$, where $\mathscr{G} = \bigcup_{i=1}^{m}\mathscr{G}_i$ and $\mathcal{A} = \bigcup_{i=1}^{m}\mathcal{A}_i$.*

Clearly, a geometrical G-system can be applied for holding on the global behavior of systems $\mathscr{G}_1,\mathscr{G}_2,\cdots,\mathscr{G}_m$. For example, a geometrical $K_4-\{e\}$-system is shown in Fig.11, where, $\mathbb{R}_i^3$, $1 \leq i \leq 4$ are Euclidean spaces with dimensional 3 and $\mathbb{R}_i^3\bigcap\mathbb{R}_j^3$ maybe homeomorphic to $\mathbb{R},\mathbb{R}^2$ or $\mathbb{R}^3$ for $1 \leq i,j \leq 4$.

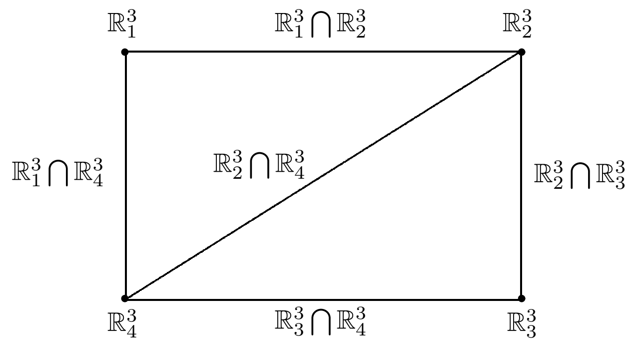


Fig.11

Problem 5.7 *Characterize geometrical G-systems $G[\mathscr{G}, \mathcal{A}]$. Particularly, characterize these geometrical G-systems, such as those of Euclidean geometry, Riemannian geometry, Lobachevshy-Bolyai-Gauss geometry for complete graphs $G = K_m$, complete k-partite graph $K_{m_1,m_2,\cdots,m_k}$, path P_m or circuit C_m.*

Problem 5.8 *Characterize geometrical G-systems $G[\mathscr{G}, \mathcal{A}]$ for topological or differentiable manifold, particularly, Euclidean space, projective space for complete graphs $G = K_m$, complete k-partite graph $K_{m_1,m_2,\cdots,m_k}$, path P_m or circuit C_m.*

It should be noted that classic geometrical system are mostly K_1-systems, such as those of Euclidean geometry, projective geometry,$\cdots$, etc., also a few K_2-systems. For example, the topological group and Lie group are in fact geometrical K_2-systems, but neither K_m-system with $m \geq 3$, nor $G \not\simeq K_m$-system.

§6. **Applications**

As we known, mathematical non-systems are generally faced up human beings in scientific fields. Even through, the mathematical combinatorics contributes an approach for holding on their global behaviors.

6.1 Economics

A *circulating economic system* is such a overall balance input-output $M(t) = \bigcup\limits_{i=1}^{m} M_i(t)$ underlying a topological graph $G[M(t)]$ that there are no rubbish in each producing department. Whence, there is a circuit-decomposition $G\left[M(t)\right] = \bigcup\limits_{i=1}^{l} \overrightarrow{C}_s$ such that each output of a producing department $M_i(t)$, $1 \leq i \leq m$ is on a directed circuit $\overrightarrow{C}_s$ for an integer $1 \leq s \leq l$, such as those shown in Fig.12.

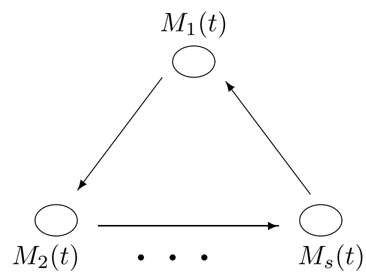


Fig.12

Assume that there are m producing departments $M_1(t)$, $M_2(t), \cdots, M_m(t)$, x_{ij} output values of $M_i(t)$ for the department $M_j(t)$ and d_i for the social demand. Let $F_i(x_{1i}, x_{2i}, \cdots, x_{ni})$ be the producing function in $M_i(t)$. Then the input-output model of a circulating economic system can be characterized by a system of equations

$$\begin{cases} F_1(\overline{x}) = \sum\limits_{j=1}^{m} x_{1j} + d_1 \\ F_2(\overline{x}) = \sum\limits_{j=1}^{m} x_{2j} + d_2 \\ \dots\dots\dots\dots\dots\dots\dots \\ F_m(\overline{x}) = \sum\limits_{j=1}^{m} x_{mj} + d_m \end{cases}$$

Generally, this system is non-solvable even if it is a linear system. Nevertheless, it is a G-system of equations. The main task is not finding its solutions, but determining whether it runs smoothly, i.e., a macro-economic behavior of system.

6.2 Epidemiology

Assume that there are three kind groups in persons at time t, i.e., infected $I(t)$, susceptible $S(t)$ and recovered $R(t)$ with $S(t) + I(t) + R(t) = 1$. Then one established the *SIR model* of infectious disease as follows:

$$\begin{cases} \frac{dS}{dt} = -kIS, \\ \frac{dI}{dt} = kIS - hI, \\ S(0) = S_0, I(0) = I_0, R(0) = 0, \end{cases},$$

which are non-linear equations of first order.

If the number of persons in an area is not constant, let $C_1, C_2, \cdots, C_m$ be m segregation areas with respective $N_1, N_2, \cdots, N_m$ persons. Assume at time t, there are $U_i(t)$, $V_i(t)$ persons moving in or away C_i. Thus $S_i(t) + I_i(t) - U_i(t) + V_i(t) = N_i$. Denoted by $c_{ij}(t)$ the persons moving from C_i to C_j for integers $1 \leq i, j \leq m$. Then

$$\sum_{s=1}^{m} c_{si}(t) = U_i(t) \quad \text{and} \quad \sum_{s=1}^{m} c_{is}(t) = V_i(t).$$

A combinatorial model of infectious disease is defined by a topological graph G following:

$$V(G) = \{C_1, C_2, \cdots, C_m\},$$
$$E(G) = \{(C_i, C_j) |\ \text{there are traffic means from } C_i \text{ to } C_j,\ 1 \le i, j \le m\},$$
$$L(C_i) = N_i,\quad L^+(C_i, C_j) = c_{ij} \text{ for } \forall (C_i, C_j) \in E\left(G^l\right),\ 1 \le i, j \le m,$$

such as those shown in Fig.13.

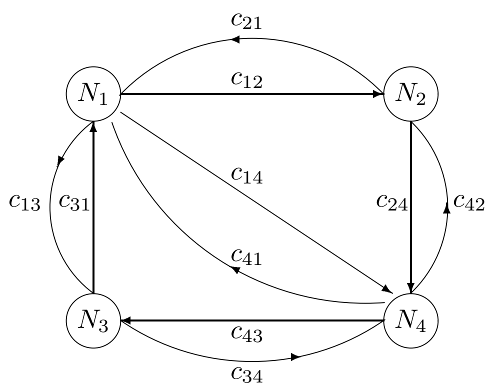


Fig.13

In this case, the SIR model for areas $C_i,\ 1 \le i \le m$ turns to

$$\left.\begin{array}{l} \dfrac{dS_i}{dt} = -kI_iS_i, \\ \dfrac{dI_i}{dt} = kI_iS_i - hI_i, \\ S_i(0) = S_{i0}, I_i(0) = I_{i0}, R(0) = 0, \end{array}\right\} 1 \le i \le m,$$

which is a non-solvable system of differential equations.

Even if the number of an area is constant, the SIR model works only with the assumption that a healed person acquired immunity and will never be infected again. If it does not hold, the SIR model will not immediately work, such as those of cases following:

Case 1. there are m known virus $\mathscr{V}_1, \mathscr{V}_2, \cdots, \mathscr{V}_m$ with infected rate k_i, heal rate h_i for integers $1 \le i \le m$ and an person infected a virus $\mathscr{V}_i$ will never infects other viruses $\mathscr{V}_j$ for $j \ne i$.

Case 2. there are m varying $\mathscr{V}_1, \mathscr{V}_2, \cdots, \mathscr{V}_m$ from a virus $\mathscr{V}$ with infected rate k_i, heal rate h_i for integers $1 \le i \le m$ such as those shown in Fig.14.

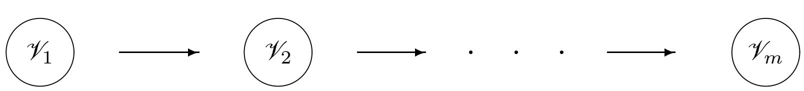


Fig.14

However, it is easily to establish a non-solvable differential model for the spread of viruses following by combining SIR model:

$$\begin{cases} \dot{S} = -k_1 SI \\ \dot{I} = k_1 SI - h_1 I \\ \dot{R} = h_1 I \end{cases} \quad \begin{cases} \dot{S} = -k_2 SI \\ \dot{I} = k_2 SI - h_2 I \\ \dot{R} = h_2 I \end{cases} \quad \cdots \quad \begin{cases} \dot{S} = -k_m SI \\ \dot{I} = k_m SI - h_m I \\ \dot{R} = h_m I \end{cases}$$

Consider the equilibrium points of this system enables one to get a conclusion ([27]) for globally control of infectious diseases, i.e., they decline to 0 finally if

$$0 < S < \sum_{i=1}^{m} h_i \Big/ \sum_{i=1}^{m} k_i \ ,$$

particularly, these infectious viruses are globally controlled if each of them is controlled in that area.

6.3 Gravitational Field

What is the true face of gravitation? Einstein's equivalence principle says that *there are no difference for physical effects of the inertial force and the gravitation in a field small enough*, i.e., considering the curvature at each point in a spacetime to be all effect of gravitation, called *geometrization of gravitation*, which finally resulted in Einstein's gravitational equations ([2])

$$R^{\mu\nu} - \frac{1}{2}Rg^{\mu\nu} + \lambda g^{\mu\nu} = -8\pi G T^{\mu\nu}$$

in $\mathbb{R}^4$, where $R^{\mu\nu} = R^{\mu\alpha\nu}_{\alpha} = g_{\alpha\beta}R^{\alpha\mu\beta\nu}$, $R = g_{\mu\nu}R^{\mu\nu}$ are the respective Ricci tensor, Ricci scalar curvature, $G = 6.673 \times 10^{-8} cm^3/gs^2$, $\kappa = 8\pi G/c^4 = 2.08 \times 10^{-48} cm^{-1} \cdot g^{-1} \cdot s^2$ and Schwarzschild spacetime with a spherically symmetric Riemannian metric

$$ds^2 = f(t)\left(1 - \frac{r_g}{r}\right)dt^2 - \frac{1}{1 - \frac{r_g}{r}}dr^2 - r^2(d\theta^2 + \sin^2\theta d\phi^2)$$

for $\lambda = 0$. However, a most puzzled question faced up human beings is *whether the dimension of the universe is really* 3? if not, *what is the meaning of one's observations*? Certainly, if the dimension≥ 4, all these observations are nothing else but a projection of the true faces on our six organs, a pseudo-truth.

For a gravitational field $\mathbb{R}^n$ with $n \geq 4$, decompose it into dimensional 3 Euclidean spaces $\mathbb{R}^3_u$, $\mathbb{R}^3_v$, $\cdots$, $\mathbb{R}^3_w$. Then there are Einstein's gravitational equations:

$$R^{\mu_u\nu_u} - \frac{1}{2}g^{\mu_u\nu_u}R = -8\pi G T^{\mu_u\nu_u},$$

$$R^{\mu_v\nu_v} - \frac{1}{2}g^{\mu_v\nu_v}R = -8\pi G T^{\mu_v\nu_v},$$

$$\cdots\cdots\cdots\cdots\cdots ,$$

$$R^{\mu_w\nu_w} - \frac{1}{2}g^{\mu_w\nu_w}R = -8\pi G T^{\mu_w\nu_w}$$

for each $\mathbb{R}^3_u$, $\mathbb{R}^3_v, \cdots$, $\mathbb{R}^3_w$, such as a K_4-system shown in Fig.15,

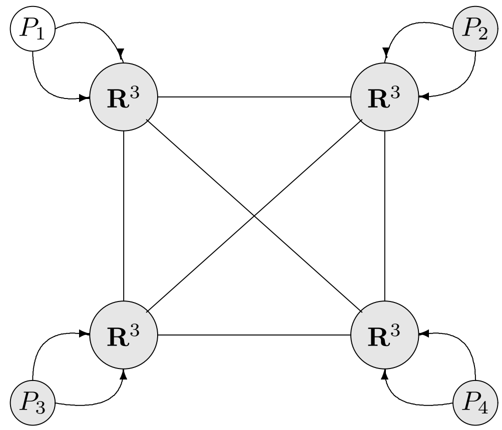


Fig.15

where P_1, P_2, P_3, P_4 are the observations. In this case, these gravitational equations can be represented by

$$R^{(\mu\nu)(\sigma\tau)} - \frac{1}{2}g^{(\mu\nu)(\sigma\tau)}R = -8\pi G T^{(\mu\nu)(\sigma\tau)}$$

with a coordinate matrix

$$[\overline{x}_p] = \begin{bmatrix} x_{11} & \cdots & x_{1\widehat{m}} & \cdots & x_{13} \\ x_{21} & \cdots & x_{2\widehat{m}} & \cdots & x_{23} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ x_{m1} & \cdots & x_{m\widehat{m}} & \cdots & x_{m3} \end{bmatrix}$$

for $\forall p \in \mathbb{R}^n$, where $\widehat{m} = \dim\left(\bigcap\limits_{i=1}^{m} \mathbf{R}^{n_i}\right)$ a constant for $\forall p \in \bigcap\limits_{i=1}^{m} \mathbf{R}^{n_i}$ and $x^{il} = \dfrac{x^l}{m}$ for $1 \leq i \leq m, 1 \leq l \leq \widehat{m}$. Then, by the *Projective Principle*, i.e., *a physics law in a Euclidean space* $\mathbb{R}^n \simeq \widetilde{\mathbb{R}} = \bigcup\limits_{i=1}^{n} \mathbb{R}^3$ *with* $n \geq 4$ *is invariant under a projection on* $\mathbb{R}^3$ *from* $\mathbb{R}^n$, one can determines its combinatorial Schwarzschild metric. For example, if $\widehat{m} = 4$, i.e., $t_\mu = t, r_\mu = r, \theta_\mu = \theta$ and $\phi_\mu = \phi$ for $1 \leq \mu \leq m$, then ([18])

$$ds^2 = \sum_{\mu=1}^{m}\left(1 - \frac{2Gm_\mu}{c^2 r}\right)dt^2 - \sum_{\mu=1}^{m}\left(1 - \frac{2Gm_\mu}{c^2 r}\right)^{-1} dr^2 - mr^2(d\theta^2 + \sin^2\theta d\phi^2)$$

and furthermore, if $m_\mu = M$ for $1 \leq \mu \leq m$, then

$$ds^2 = \left(1 - \frac{2GM}{c^2 r}\right)mdt^2 - \left(1 - \frac{2GM}{c^2 r}\right)^{-1} mdr^2 - mr^2(d\theta^2 + \sin^2\theta d\phi^2),$$

which is the most enjoyed case by human beings. If so, all the behavior of universe can be realized finally by human beings. But if $\widehat{m} \leq 3$, there are infinite underlying connected graphs, one can only find an approximating theory for the universe, i.e., "Name named is not the eternal Name", claimed by Lao Zi.

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