MEAN VALUE OF THE ADDITIVE ANALOGUE OF SMARANDACHE FUNCTION

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Abstract For any positive integer n, let $Sdf(n)$ denotes the Smarandance double factorial function, then $Sdf(n)$ is defined as least positive integer m such that $m!!$ is divisible by n. In this paper, we study the mean value properties of the additive analogue of $Sdf(n)$ and give an interesting mean value formula for it.

Keywords: Smarandance function; Additive Analogue; Mean value formula

§1. Introduction and result

For any positive integer n, let $Sdf(n)$ denotes the Smarandance double factorial function, then $Sdf(n)$ defined the least positive integer n such that $m!!$ is divisible by n, where

$$m!! = \begin{cases} 2 \cdot 4 \cdots m, & \text{if } 2|m; \\ 1 \cdot 3 \cdots m, & \text{if } 2\dagger\text{m}. \end{cases}$$

In reference [2], Professor Jozsef Sandor defined the following analogue of Smarandance double factorial function as:

$$Sdf_1(2x) = \min\{2m \in N : 2x \leq (2m)!!\}, x \in (1, \propto),$$

$$Sdf_1(2x+1) = \min\{2m+1 \in N : (2x+1) \leq (2m+1)!!\}, x \in (1, \propto),$$

which is defined on a subset of real numbers. Clearly $Sdf_1(n) = m$ if $x \in ((m-2)!!, m!!]$ for $m \geq 2$, therefore this function is defined for $x \geq 1$.

About the arithmetical properties of $Sdf(n)$, many people had ever studied it. But for the mean value properties of $Sdf_1(n)$, it seems that no one have studied before. The main purpose of this paper is to study the mean value properties of $Sdf_1(n)$, and obtain an interesting mean value formula for it. That is, we shall prove the following:

Theorem. *For any real number $x \geq 2$, we have the asymptotic formula*

$$\sum_{n \leq x} Sdf_1(n) = \frac{2x \ln x}{\ln \ln x} + O\left(\frac{x(\ln x)(\ln \ln \ln x)}{(\ln \ln x)^2}\right).$$

§2. Proof of the theorem

In this section, we shall complete the proof of the theorem. First we need the following one simple Lemma. That is,

Lemma. *For any fixed positive integer m and n with $(m-2)!! < n \le m!!$, we have the asymptotic formula*

$$m = \frac{2\ln n}{\ln\ln n} + O\left(\frac{(\ln n)(\ln\ln\ln n)}{(\ln\ln n)^2}\right).$$

Proof. To complete the proof the Lemma, we separate it into two cases:

(I) If $m = 2u$, we have $(2u-2)!! < n \le (2u)!!$. Taking the logistic computation in the two sides of the inequality, we get

$$(u-1)\ln 2 + \sum_{i=1}^{u-1} \ln i < \ln n \le u\ln 2 + \sum_{i=1}^{u} \ln i. \tag{1}$$

Then using the Euler's summation formula we have

$$\sum_{i=1}^{u} \ln i = \int_1^u \ln t dt + \int_1^u (t-[t])(\ln t)' dt = u\ln u - u + O(\ln u) \tag{2}$$

and

$$\sum_{i=1}^{u-1} \ln i = \sum_{i=1}^{u} \ln i + O(\ln u) = u\ln u - u + O(\ln u). \tag{3}$$

Combining (1), (2) and (3), we can easily deduce that

$$\ln n = u\ln u + (\ln 2 - 1)u + O(\ln u). \tag{4}$$

So

$$u = \frac{\ln n}{\ln u + (\ln 2 - 1)} + O(1). \tag{5}$$

Similarly, we continue taking the logistic computation in two sides of (5), then we also have

$$\ln u = \ln\ln n + O(\ln\ln u) \tag{6}$$

and

$$\ln\ln u = O(\ln\ln\ln n). \tag{7}$$

Hence, by (5), (6) and (7) we have

$$u = \frac{\ln n}{\ln\ln n} + O\left(\frac{(\ln n)(\ln\ln\ln n)}{(\ln\ln n)^2}\right).$$

This completes the proof of the first case.

(II) If $m = 2u+1$, we have $(2u-1)!! < n \leq (2u+1)!!$. Taking the logistic computation in the two sides of the inequality, we get

$$\sum_{i=1}^{2u} \ln i - (u\ln 2 + \sum_{i=1}^{u} \ln i) < \ln n \leq \sum_{i=1}^{2u+1} \ln i - (u\ln 2 + \sum_{i=1}^{u} \ln i). \tag{8}$$

Then using the Euler's summation formula we have

$$\sum_{i=1}^{2u} \ln i = \int_1^{2u} \ln t dt + \int_1^{2u} (t-[t])(\ln t)' dt = 2u\ln u + 2(\ln 2 - 1)u + O(\ln u) \tag{9}$$

and

$$\sum_{i=1}^{2u+1} \ln i = \sum_{i=1}^{2u} \ln i + O(\ln 2u+1) = 2u\ln u + 2(\ln 2-1)u + O(\ln u). \tag{10}$$

From (2), (3), (8), (9) and (10) we have

$$\ln n = u\ln u + (\ln 2 - 1)u + O(\ln u).$$

Therefore, we may obtain (5).

Using the similar method on the above, we may have

$$u = \frac{\ln n}{\ln\ln n} + O\left(\frac{(\ln n)(\ln\ln\ln n)}{(\ln\ln n)^2}\right).$$

This completes the proof of the second case.

Combining the above two cases, we can easily get

$$m = \frac{2\ln n}{\ln\ln n} + O\left(\frac{(\ln n)(\ln\ln\ln n)}{(\ln\ln n)^2}\right).$$

This completes the proof of Lemma.

Now we use the above Lemma to complete the proof of Theorem. For any real number $x \geq 2$, by the definition of $Sdf_1(n)$ and the above Lemma we have

$$\begin{aligned} \sum_{n\leq x} Sdf_1(n) &= \sum_{\substack{n\leq x \\ (m-2)!!<n\leq m!!}} m \\ &= \sum_{n\leq x}\left(\frac{2\ln n}{\ln\ln n} + O\left(\frac{(\ln n)(\ln\ln\ln n)}{(\ln\ln n)^2}\right)\right) \\ &= 2\sum_{n\leq x}\frac{\ln n}{\ln\ln n} + O\left(\frac{x(\ln x)(\ln\ln\ln x)}{(\ln\ln x)^2}\right). \end{aligned} \tag{11}$$

By the Euler's summation formula, we deduce that

$$\begin{aligned}\sum_{n\le x}\frac{\ln n}{\ln\ln n} &= \int_2^x\frac{\ln t}{\ln\ln t}dt+\int_2^x(t-[t])(\frac{\ln t}{\ln\ln t})'dt+\frac{\ln x}{\ln\ln x}(x-[x])\\ &= \frac{x\ln x}{\ln\ln x}+O\left(\frac{x}{\ln\ln x}\right). \quad (12)\end{aligned}$$

Therefore, from (11) and (12) we have

$$\sum_{n\le x}Sdf_1(n)=\frac{2x\ln x}{\ln\ln x}+O\left(\frac{x(\ln x)(\ln\ln\ln x)}{(\ln\ln x)^2}\right).$$

This completes the proof of Theorem.

References

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