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Minimum Cycle Base of Graphs Identified by Two Planar Graphs

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Abstract: In this paper, we study the minimum cycle base of the planar graphs obtained from two 2-connected planar graphs by identifying an edge (or a cycle) of one graph with the corresponding edge (or cycle) of another, related with map geometries, i.e., Smarandache 2-dimensional manifolds. Also, we give a formula for calculating the length of minimum cycle base of a planar graph $N(d, \lambda)$ defined in paper [11].

Key Words: graph, planar graph, cycle space, minimum cycle base.
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§1. Introduction

Throughout this paper we consider simple and undirected graphs. The cardinality of a set A is $|A|$. Let's begin with some terminologies and some facts about cycle bases of graphs. Let $G(V, E)$ be a 2-connected graph with vertex set V and edge set E. The set $\mathcal{E}$ of all subsets of E forms an $|E|$-dimensional vector space over $GF(2)$ with vector addition $X \oplus Y = (X \cup Y) \setminus (X \cap Y)$ and scalar multiplication $1 \bullet X = X, 0 \bullet X = \emptyset$ for all $X, Y \in \mathcal{E}$. A *cycle* is a connected graph whose any vertex degree is 2. The set $\mathcal{C}$ of all cycles of G forms a subspace of $(\mathcal{E}, \oplus, \bullet)$ which is called the *cycle space* of G. The dimension of the cycle space $\mathcal{C}$ is the Betti number of G, say $\beta(G)$, which is equal to $|E(G)| - |V(G)| + 1$. A base $\mathcal{B}$ of the cycle space of G is called a *cycle base* of G.

The length $|C|$ of a cycle C is the number of its edges. The length $l(\mathcal{B})$ of a cycle base $\mathcal{B}$ is the sum of lengths of all its cycles. A *minimum cycle base* (or MCB in short) is a cycle base with minimal length. A graph may has many minimum cycle bases, but every two minimum cycle bases have the same length.

Let G be a 2-connected planar graph embedded in the plane. G has $|E(G)| - |V(G)| + 2$ faces by Euler formula. There is exactly one face of G being unbounded which is called the

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exterior of G. All faces but the exterior of G are called interior faces of G. Each interior face of G has a cycle as its boundary which is called an *interior facial cycle*. Also, the cycle of G being incident with the exterior of G is called *the exterior facial cycle*.

We know that if G is a 2-connected planar graph embedded in the plane, then any set of $|E(G)| - |V(G)| + 1$ facial cycles forms a cycle base of G. For a 2-connected planar graph, we ask whether there is a minimum cycle base such that each cycle is a facial cycle. The answer isn't confirmed. The counterexample is easy to be constructed by Lemma 1.1. Need to say that Lemma 1.1 is a special case of Theorem A in the reference [10] which is deduced by Hall Theorem.

Lemma 1.1 *Let $\mathcal{B}$ be a cycle base of a 2-connected graph G. Then $\mathcal{B}$ is a minimum cycle base of G if and only if for any cycle C of G and cycle B in $\mathcal{B}$, if $B \in Int(C)$, then $|C| \geq |B|$, where Int (C) denotes the set of cycles in $\mathcal{B}$ which generate C.*

For some special 2-connected planar graph, there exist a minimum cycle base such that each cycle is a facial cycle. For example, Halin graph and outerplanar graph are such graphs. A *Halin graph* $H(T)$ consists of a tree T embedded in the plane without subdivision of an edge together with the additional edges joining the 1-valent vertices consecutively in their order in the planar embedding. It is clear that a Halin graphs is a 3-connected planar graph. The exterior facial cycle is called *leaf-cycle*.

Lemma 1.2[9,12] *Let $H(T)$ be a Halin graph embedded in the plane such that the leaf-cycle is the exterior facial cycle. Let $\mathcal{F}$ denote the set of interior facial cycles of $H(T)$. Then $\mathcal{F}$ is a minimum cycle base of $H(T)$.*

A planar graph G is outerplanar if it can be embedded in the plane such that all vertices lie on the exterior facial cycle C.

Lemma 1.3[6,9] *Let $G(V, E)$ be a 2-connected outerplanar graph embedded in the plane with C as its exterior facial cycle. Let $\mathcal{F}$ be the set of interior facial cycles. Then $\mathcal{F}$ is the minimum cycle base of G, and $l(\mathcal{F}) = 2|E| - |V|$.*

Apart from the above mentioned minimum cycle bases of a Halin graph and an outerplanar graph, many peoples researched minimum cycle bases of graphs. H. Ren et al. [9] not only gave a sufficient and necessary condition for minimum cycle base of a 2-connected planar graph, but also studied minimum cycle bases of graphs embedded in non-spherical surfaces and presented formulae for length of minimum cycle bases of some graphs such as the generalized Petersen graphs, the circulant graphs, etc. W.Imrich et al. [4] studied the minimum cycle bases for the cartesian and strong product of two graphs. P.Vismara [13] discussed the union of all the minimum cycle bases of a graph. What about the minimum cycle base of the graph obtained from two 2-connected planar graphs by identifying some corresponding edges? This problem is related with *map geometries*, i.e., *Smarandache 2-dimensional manifolds* (see [8] for details). We will consider it in this paper.

§2. MCB of graphs obtained by identifying an edge of planar graphs

Let G_1 and G_2 be two graphs and P_i be a path (or a cycle) in G_i for $i = 1, 2$. Suppose the length of P_1 is same as that of P_2. By identifying P_1 with P_2, we mean that the vertices of P_1 are identified with the corresponding vertices of P_2 and the multiedges are deleted.

Theorem 2.1 *Let G_1 and G_2 be two 2-connected planar graphs embedded in the plane. Let e_i be an edge in $E(G_i)$ such that e_i is in the exterior facial cycle of G_i for $i = 1, 2$. Let G be the graph obtained from G_1 and G_2 by identifying e_1 and e_2 such that G_2 is in the exterior of G_1. If the set of interior facial cycles of G_i, say $\mathcal{F}_i$, is a minimum cycle base of G_i for $i = 1, 2$, then $\mathcal{F}_1 \cup \mathcal{F}_2$ is a minimum cycle base of G.*

Proof Obviously, the graph G is a 2-connected planar graph and each cycle of $\mathcal{F}_1 \cup \mathcal{F}_2$ is a facial cycle of G. Since $|E(G)| = |E(G_1)|+|E(G_2)|-1$ and $|V(G)| = |V(G_1)|+|V(G_2)|-2$, G has $|E(G)|-|V(G)|+2 = (|E(G_1)|-|V(G_1)|+1)+(|E(G_2)|-|V(G_2)|+1)+1 = |\mathcal{F}_1|+|\mathcal{F}_2|+1$ faces. So $|\mathcal{F}| = |\mathcal{F}_1| + |\mathcal{F}_2| = |E(G)| - |V(G)| + 1$, and $\mathcal{F}$ is a cycle base of G.

Now we prove that $\mathcal{F}$ is a minimum cycle base of G. Suppose F is a cycle of G and $F = f_1 \oplus f_2 \oplus \cdots \oplus f_q$, where $f_j \in \mathcal{F}$ for $j = 1, 2, \cdots, q$. By Lemma 1.1, We need to prove $|F| \geq |f_j|$ for $j = 1, 2, \cdots, q$.

If $E(F) \subset E(G_1)$ (or $E(G_2)$), then f_j is in $\mathcal{F}_1$ (or $\mathcal{F}_2$) for $j = 1, 2, \cdots, q$. By the fact that $\mathcal{F}_i$ is a minimum cycle base of G_i for $i = 1, 2$ and Lemma 1.1, $|F| \geq |f_j|$ for $j = 1, 2, \cdots, q$.

Let e be the edge of G obtained by e_1 identified with e_2. Suppose $e = \{uv\}$. If edges of F aren't in G_1 entirely, then F must pass through u and v. So $e \cup F$ can be partitioned into two cycles, say F_1 and F_2. Suppose $E(F_i) \subset E(G_i)$ for $i = 1, 2$. Then $|F| > |F_i|$ for $i = 1, 2$. Suppose $F_1 = f_1 \oplus f_2 \oplus \cdots \oplus f_p$ and $F_2 = f_{p+1} \oplus f_{p+2} \oplus \cdots \oplus f_q$. By the fact that $\mathcal{F}_i$ is a minimum cycle base of G_i for $i = 1, 2$ and Lemma 1.1, $|F| > |F_1| \geq |f_i|$ for $i = 1, 2, \cdots, p$ and $|F| > |F_2| \geq |f_i|$ for $i = p+1, p+2, \cdots, q$.

Thus we complete the proof. □

Applying Theorem 2.1 and the induction principle, it is easy to prove the following conclusion.

Corollary 2.1 *Let $G_1, G_2, \cdots, G_k$ be $k(k \geq 3)$ 2-connected planar graphs embedded in the plane. Let e_i be an edge in $E(G_i)$ such that e_i is in the exterior facial cycle of G_i for $i = 1, 2, \cdots, k$. Let G'_1 be the graph obtained from G_1 and G_2 by identifying e_1 with e_2 such that G_2 is in the exterior of G_1, Let G'_2 be the graph obtained from G'_1 and G_3 by identifying e_3 with some edge in the exterior face of G'_1 such that G_3 is in the exterior of G'_1, and so on. Let G be the last obtained graph in the above process. If the set of interior facial cycles of G_i, say $\mathcal{F}_i$, is a minimum cycle base of G_i for $i = 1, 2, \cdots, k$, then $\cup_{i=1}^{k}\mathcal{F}_i$ is a minimum cycle base of G.*

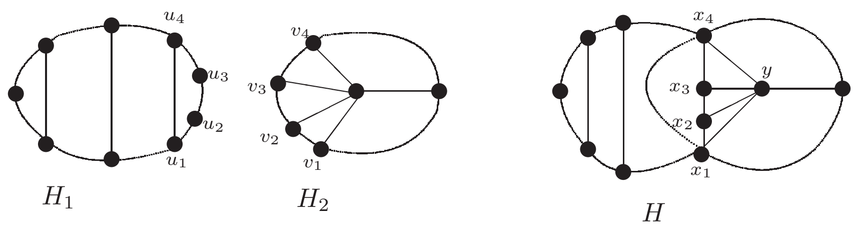

Fig.2.1

Remark: In Theorem 2.1, if e_1 is replaced by a path with length at least two and e_2 by the corresponding path, then the conclusion of the theorem doesn't hold. We consider the graph H shown in Fig.2.1, where H is obtained from H_1 and H_2 by identified $P_1 = u_1u_2u_3u_4$ with $P_2 = v_1v_2v_3v_4$. For the graph H, let $C = x_1x_2x_3x_4x_1$ and $D = x_1yx_4x_1$. Since $|C| > |D|$, the set of interior facial cycle of H isn't its minimum cycle base by Lemma 1.1.

Furthermore, if e_1 is replaced by a cycle and e_2 by the corresponding cycle in Theorem 2.1, then the conclusion of Theorem isn't true. The counterexample is easy to construct, which is left to readers. But if G_1 is a special planar graph, similar results to Theorem 2.1 will be shown in the next section.

§3. MCB of graphs obtained by identifying a cycle of planar graphs

An $r \times s$ cylinder is the graph with r radial lines and s cycles, where $r \geq 0, s > 0$. A 4×3 cylinder is shown in Fig.3.1. The innermost cycle is called the *central cycle*. $r \times s$ cylinder take an important role in discussion of the minor of planar graph with sufficiently large tree-width in paper[10].

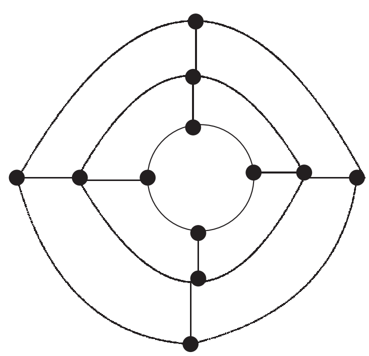

Fig.3.1

Theorem 3.1 *Let G_1 be an $r \times s (r \geq 4)$ cylinder embedded in the plane such that C is its central cycle. Let G_2 be a planar graph embedded in the plane such that the exterior facial cycle D has the same vertices as that of C. Let G be the graph obtained from G_1 and G_2 by identifying C and D such that G_2 is in the interior of G_1. If the set of interior facial cycles of G_2, say $\mathcal{F}_2$, is its a minimum cycle base, then the set of interior facial cycles of G, say $\mathcal{F}$, is a minimum cycle base of G.*

Proof At first, $\mathcal{F}$ is a cycle base of G. We need prove $\mathcal{F}$ is minimal.

Let $\mathcal{F}_1 = \mathcal{F}\backslash\mathcal{F}_2$. Obviously, each element of $\mathcal{F}_1$ has length 4. Suppose F is a cycle of G and $F = f_1 \oplus f_2 \oplus \cdots \oplus f_q$, where $f_j \in \mathcal{F}$ for $j = 1, 2, \cdots, q$. If we prove $|F| \geq |f_j|$ for $j = 1, 2, \cdots, q$, then $\mathcal{F}$ is a minimum cycle base of G by Lemma 1.1.

Let R be the open region bounded by F, and R' be the open region bounded by C (or D) of G_1 (or G_2). We consider the following four cases.

Case 1 $R' \cap R = \emptyset$.

Then F is a cycle of G_1 and F is generated by $\mathcal{F}_1$. Since the girth of G_1 is 4, $|F| \geq |f_j| = 4$ for $j = 1, 2, \cdots, q$.

Case 2 $R' \subset R$.

Then $|F| \geq |C| \geq 4$, because the number of radial lines which F crosses can't be less than the number of vertices of C. For a fixed f_j, if it is in the interior of C then $|f_j| \leq |C| \leq |F|$ by Lemma 1.1, because $\mathcal{F}_2$ is a minimum cycle base of G_2. If f_j is in the exterior of C, then $|f_j| = 4$. So $|f_i| \leq |F|$ for $j = 1, 2, \cdots, q$.

Case 3 $R \subset R'$.

Then F is a cycle of G_2. By Lemma 1.1, $|F| \geq |f_j|$ for $j = 1, 2, \cdots, q$.

Case 4 $R' \cap R \neq \emptyset$ and R' is not in the interior of R.

Then F must has at least one edge in $E(G_2)\backslash E(C)$ and at least three edges in $E(G_1)$. So $|F| \geq 4$. No loss of generality, suppose $f_1, f_2, \cdots, f_p$ are cycles of $\{f_1, f_2, \cdots, f_q\}$ that are in the exterior of C. Since $|f_j| = 4$, $|F| \geq |f_j|$ for $j = 1, 2, \cdots, p$.

Next we prove $|F| \geq |f_j|$ for $j = p + 1, p + 2, \cdots, q$, where f_j is in the interior of C.

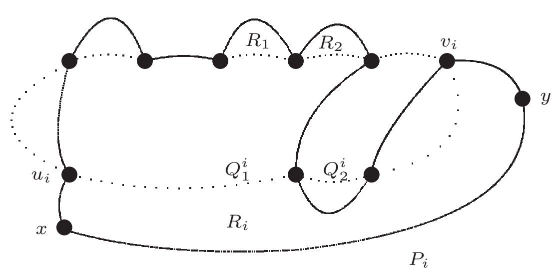

Fig. 3.2

Let $R" = R \backslash (R' \cap R)$. $R"$ may be the union of several regions. Let $R" = R_1 \cup R_2 \cup \cdots \cup R_l$ satisfying the condition that $R_i \cap R_j$ is empty or a point for $i \neq j, 1 \leq i, j \leq l$. Let B_i be the boundary of R_i for $i = 1, 2, \cdots, l$. Then B_i is a cycle in the exterior of C. For a fixed B_i, there may be many vertices of B_i in $V(F) \cap V(C)$, which can be found in Fig.3.2. We select two vertices u_i and v_i of B_i satisfying the following conditions:

(1) u_i and v_i are in C;

(2) there is a path of B_i, say P_i, such that its endvertices are u_i and v_i and P_i is in the exterior of C;

(3) if M_i is the path of B_i deleted $E(P_i)$, and if M_i' is the path of C such that its endvertices are u_i and v_i and M_i' is internally disjoint from B_i, then M_i is in the interior of the cycle which is the union of M_i' and P_i.

Note that M_i may contains many disjoint paths of C, suppose they are $Q_1^i, Q_2^i, \cdots, Q_t^i$. Let x, y be two vertices in P_i, which are adjacent to u_i, v_i respectively.

Obviously, x, y are in G_1. Let P_i' be the subpath of P_i between x and y. Considering the number of radial lines (including radial line x, y lie on) which P_i' crosses is not less than the number of vertices of $\cup_{j=1}^{t} Q_j^i$, $|P_i| > |P_i'| \geq \sum_{j=1}^{t} |Q_j^i|$.

Since $R' \cap R$ may be the union of some regions, we suppose $R' \cap R = D_1 \cup D_2 \cup \cdots \cup D_s$. Let $A_1, A_2, \cdots, A_s$ be boundaries of $D_1, D_2, \cdots, D_s$ respectively. For a fixed A_i, its edges may be partitioned into two groups, one containing edges of F, denoted as A_i^F, another containing edges of C, denoted as A_i^C. Then

$$\begin{aligned}\sum_{i=1}^{s} |A_i| &= \sum_{i=1}^{s} |A_i^F| + \sum_{i=1}^{s} |A_i^C| \\ &= \sum_{i=1}^{s} |A_i^F| + \sum_{i=1}^{l} \sum_{j=1}^{t} |Q_j^i| \\ &< \sum_{i=1}^{s} |A_i^F| + \sum_{i=1}^{s} |P_i| \\ &< |F|\end{aligned}$$

Hence $|F| > |A_i|$ for $i = 1, 2, \cdots, s$. Since any A_i is a cycle of G_2 and $\mathcal{F}_1$ is a minimum cycle base of G_2, $|A_i| \geq |f_j|$ for $j = i_1, i_2, \cdots, i_n$, by lemma 2.1, where $\{i_1, i_2, \cdots, i_n\} \subset \{p+1, p+2, \cdots, q\}$. Hence, $|F| > |f_{p+j}|$ for $i = 1, 2, \cdots, q-p$.

By the previous discussion and Lemma 1.1, $\mathcal{F}$ is a minimum cycle base of G. □

Since the minimum cycle base of a cycle is itself, a minimum cycle base of an $r \times s (r \geq 4)$ cylinder embedded in the plane is the set of its interior facial cycles by Theorem 3.1, and the length of its MCB is $r + 4r(s-1) = r(4s-3)$.

By Lemmas 1.2, 1.3 and Theorem 3.1. we get two corollaries following.

Corollary 3.1 *Assume an $r \times s (r \geq 4)$ cylinder, a Halin graph $H(T)$ are embedded in the plane with C the central cycle and C' the leaf-cycle of $H(T)$ containing the same vertices as C, respectively. Let G be the graph obtained from the $r \times s$ cylinder and $H(T)$ by identifying C and C' such that $H(T)$ is in the interior of the $r \times s$ cylinder. Then a minimum cycle base of G is the set of interior facial cycles of G.*

Corollary 3.2 *Assume an $r \times s (r \geq 4)$ cylinder, a 2-connected outplanar graph H be embedded in the plane with C the central cycle and C' the exterior facial cycle containing same vertices as C of H containing the same vertices as C, respectively. Let G be the graph obtained from the $r \times s$ cylinder and H by identifying C and C' such that H is in the interior of the $r \times s$ cylinder. Then a minimum cycle base of G is the set of interior facial cycles of G. Furthermore, the length of a MCB of G is $r(4s-5) + 2|E(H)|$.*

Proof Let $\mathcal{F}$ be the set of interior facial cycles of G. By Theorem 3.1, $\mathcal{F}$ is a minimum cycle base of G. $\mathcal{F}$ can be partitioned into two groups $\mathcal{F}_1$ and $\mathcal{F}_2$, where $\mathcal{F}_1$ is the set of interior facial cycles of H and $\mathcal{F}_2$ the set of 4-cycles. Then the length of a MCB of G is $l(\mathcal{F}) = l(\mathcal{F}_1) + l(\mathcal{F}_2) = 4r(s-1) + 2|E(H)| - |V(H)| = (4s-5)r + 2|E(H)|$. □

As application of Corollary 3.1, we find a formula for the length of minimum cycle base of a planar graph $N(d,\lambda)$, which can be found in paper[10].

When $\lambda \geq 1$ is an integer, the graph Y_λ is tree as shown in Fig.3.3. Thus Y_λ has $3\times 2^{\lambda-1}$ 1-valent vertices and Y_λ has $3\times 2^{\lambda}-2$ vertices. If 1-valent vertices of Y_λ are connected in their order in the planar embedding, we obtain a special Halin graph, denoted by $H(\lambda)$.

Suppose a $(3\times 2^{\lambda-1})\times d$ cylinder is embedded in the plane such that its central cycle C has $3\times 2^{\lambda-1}$ vertices. The graph obtained from $(3\times 2^{\lambda-1})\times d$ cylinder and $H(\lambda)$ with leaf-cycle C' containing $3\times 2^{\lambda-1}$ vertices by identifying C and C' such that $H(\lambda)$ is in the interior of $(3\times 2^{\lambda-1})\times d$ cylinder is denoted as $N(d,\lambda)$. N. Roberterson and P.D. Seymour[10] proved that for all $d\geq 1, \lambda\geq 1$ the graph $N(d,\lambda)$ has tree-width $\leq 3d+1$.

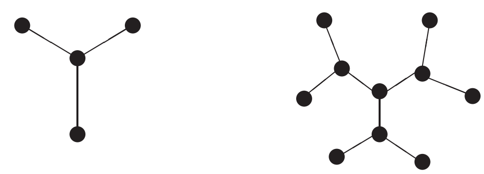

Fig.3.3

Theorem 3.2 *The length of minimum cycle base of $N(d,\lambda)$($\lambda\geq 2$) is* $3(d-1)\times 2^{\lambda+1} + 9\times 2^{\lambda} - 3\times 2^{\lambda-1} - 6$.

Proof Let $\mathcal{F}$ be the set of interior facial cycles of $N(d,\lambda)$. Then $\mathcal{F}$ is a minimum cycle base of $N(d,\lambda)$ by Corollary 3.1.

Let $\mathcal{F}_1$ be a subset of $\mathcal{F}$ which is the set of interior facial cycles of $N(1,\lambda)$ (a Halin graph). Then $\mathcal{F}_1$ consists of 3 $(2\lambda+1)$-cycles and 3×2^j $(2\lambda-2j-1)$-cycles for $j=0,1,2,\cdots,\lambda-2$.

Let $\mathcal{F}_2 = \mathcal{F}\setminus\mathcal{F}_1$. Then each cycle of $\mathcal{F}_2$ has length 4. Since the leaf-cycle of $N(1,\lambda)$ has $3\times 2^{\lambda-1}$ vertices, there are $3(d-1)\times 2^{\lambda-1}$ 4-cycles in $\mathcal{F}_2$ all together. The length of $\mathcal{F}$ is

$$\begin{aligned}
l(\mathcal{F}) &= \sum_{j=0}^{\lambda-2} 3\times 2^{j-1}(2\lambda-2j-1) + 3(2\lambda+1) + 4\times 3(d-1)\times 2^{\lambda-1} \\
&= 3[\sum_{j=0}^{\lambda-2}\lambda 2^{j+1} - 2\sum_{j=0}^{\lambda-2} j2^j - \sum_{j=0}^{\lambda-2} 2^j] + (6\lambda+3) + 3(d-1)\times 2^{\lambda+1} \\
&= 3[(\lambda 2^{\lambda} - 2\lambda) - 2(\lambda-3)2^{\lambda-1} - 4 - 2^{\lambda-1} + 1] \\
&+ (6\lambda+3) + 3(d-1)\times 2^{\lambda+1} \\
&= 3(d-1)\times 2^{\lambda+1} + 9\times 2^{\lambda} - 3\times 2^{\lambda-1} - 6
\end{aligned}$$

Hence, the length of minimum cycle base of $N(d,\lambda)$ is $3(d-1)\times 2^{\lambda+1} + 9\times 2^{\lambda} - 3\times 2^{\lambda-1} - 6$.□

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