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The Smarandache minimum and maximum functions

József Sándor

Babes-Bolyai University of Cluj, Romania

Abstract This papers deals with the introduction and preliminary study of the Smarandache minimum and maximum functions.
Keywords Smarandache minimum and maximum functions; arithmetical properties.

1. Let $f\colon N^* \longrightarrow N$ be a given arithmetic function and $A \subset N$ a given set. The arithmetic function

$$F_f^A(n) = \min\{k \in A : n \mid f(k)\} \tag{1}$$

has been introduced in [4] and [5].

For $A = N, f(k) = k!$ one obtains the Smarandache function; For $A = N^*, A = p = \{2, 3, 5, \cdot\}$ = set of all primes, one obtains a function

$$P(n) = \min\{k \in P : n \mid k!\} \tag{2}$$

For the properties of this function, see [4] and [5]. The "dual" function of (1) has been defined by

$$G_g^A(n) = \max\{k \in A : g(k) \mid n\}, \tag{3}$$

where $g : N^* \longrightarrow N$ is a given function, and $A \in N$ is a given set. Particularly, for$A = N^*, g(k) = k!$, one obtains the dual of the Smarandache function,

$$S_*(n) = \max\{k \geq 1 : k! \mid n\} \tag{4}$$

For the properties of this function, see [4] and [5]. F.Luca [3], K.Atanassov [1] and L.le [2] have proved in the affirmative a conjecture of the author.
For $A = N^*$ and $f(k) = g(k) = \varphi(k)$ in (1), resp.(3) one obtains the Euler minimum, resp. maximum-function, defined by

$$E(n) = \min\{k \geq 1 : n \mid \varphi(k)\}, \tag{5}$$

$$E_*(n) = \max\{k \geq 1 : \varphi(k) \mid n\} \tag{6}$$

For the properties of these function, see [6]. When $A = N^*, f(k) = d(k)$ =number of divisors of k, one obtains the divisor minimum function (see [4], [5] and [7])

$$D(n) = \min\{k \geq 1 : n \mid d(k)\}. \tag{7}$$

It is interesting to note that the divisor maximum function (i.e., the " dual" of $D(n)$) given by

$$D_*(n) = \max\{k \geq 1 : d(k) \mid n\} \tag{8}$$

is not well defined! Indeed, for any prime p one has $d(p^{n-1}) = n \mid n$ and p^{n-1} is unbounded as $p \longrightarrow \infty$. For a finite set A, however $D_*^A(n)$ does exist. On one hand, it has been shown in [4] and [5] that

$$\sum(n) = \min\{k \geq 1 : n \mid \sigma(k)\} \tag{9}$$

(denoted there by $F_\sigma(n)$) is well defined. (Here $\sigma(k)$ denotes the sum of all divisors of k). The dual of the sum-of-divisors minimum function is

$$\sum{}_*(n) = \max(k \geq 1 : \sigma(k) \mid n\}) \tag{10}$$

Since $\sigma(1) = 1 \mid n$ and $\sigma(k) \geq k$, clearly $\sum_*(n) \leq n$, so this function is well defined (see [8]).

2. The Smarandache minimum function will be defined for $A = N^*, f(k) = S(k)$ in (1). Let us denote this function by $S_{\min}$:

$$S_{\min}(n) = \min\{k \geq 1 : n \mid S(k)\} \tag{11}$$

Let us assume that $S(1) = 1$, i. e., $S(n)$ is defined by (1) for $A = N^*, f(k) = k!$:

$$S(n) = \min\{k \geq 1 : n \mid k!\} \tag{12}$$

Otherwise (i.e.when $S(1) = 0$) by $n \mid 0$ for all n, by (11) for one gets the trivial function $S_{min}(n) = 0$. By this assumption, however, one obtains a very interesting (and difficult) function s_{min} given by (11). Since $n \mid S(n!) = n$, this function is correctly defined.

The Smarandache maximum function will be defined as the dual of S_{min} :

$$S_{\max}(n) = \max(k \geq 1 : S(k) \mid n\}. \tag{13}$$

We prove that this is well defined. Indeed, for a fixed n, there are a finite number of divisors of n, let $i \mid n$ be one of them. The equation

$$S(k) = i \tag{14}$$

is well-known to have a number of $d(i!) - d((i-1)!)$ solutions, i. e., in a finite number. This implies that for a given n there are at most finitely many k with $S(k) \mid k$, so the maximum in (13) is attained.

Clearly $S_{\min}(1) = 1, S_{\min}(2) = 2, S_{\min}(3) = 3, S_{\min}(4) = 4, S_{\min}(5) = 5, S_{\min}(6) = 9, S_{\min}(7) = 7, S_{\min}(8) = 32, S_{\min}(9) = 27, S_{\min}(10) = 25, S_{\min}(11) = 11$, etc, which can be determined from a table of Smarandache numbers:

n	1	2	3	4	5	6	7	8	9	10	11	12	13
S(n)	1	2	3	4	5	3	7	4	6	5	11	4	13

n	14	15	16	17	18	19	20	21	22	23	24	25
S(n)	7	5	6	7	6	19	5	7	11	23	4	10

We first prove that:

Theorem 1. $S_{\min}(n) \geq n$ for all $n \geq 1$, with equality only for

$$n = 1, 4, p (p = \text{prime}) \tag{15}$$

Proof. Let $n \mid S(k)$. If we would have $k < n$, then since $S(k) \leq k < n$ we should get $S(k) < n$, in contradiction with $n \mid S(k)$. Thus $k \geq n$, and taking minimum, the inequality follows. There is equality for $n = 1$ and $n = 4$. Let now $n > 4$. If $n = p$ =prime, then $p \mid S(p) = p$, but for $k < p, p \dagger S(k)$. Indeed, by $S(k) \leq k < p$ this is impossible. Reciprocally, if $min\{k \geq 1 : n \mid S(k)\} = n$, then $n \mid S(n)$, and by $S(n) \leq n$ this is possible only when $S(n) = n$, i. e., when $n = 1, 4, p(p = \text{prime})$.

Theorem 2. For all $n \geq 1$,

$$S_{\min}(n) \leq n! \leq S_{\max}(n) \tag{16}$$

Proof. Since S(n!)=n, definition (11) gives the left side of (16), while definition (13) gives the right side inequality.

Corollary. The series $\sum_{n \geq 1} \frac{1}{S_{\min}(n)}$ is divergent, while the series $\sum_{n \geq 1} \frac{1}{S_{\max}(n)}$ is convergent.

Proof. Since $\sum_{n \geq 1} \frac{1}{S_{\max}(n)} \leq \sum_{n \geq 1} \frac{1}{n!} = e - 1$ by (16), this series is convergent. On the other hand,

$$\sum_{n \geq 1} \frac{1}{S_{\min}(n)} \geq \sum_{p} \frac{1}{S_{min}(p)} = \sum_{p} \frac{1}{p} = +\infty,$$

so the first series is divergent.

Theorem 3. For all primes p one has

$$S_{\max}(p) = p! \tag{17}$$

Proof. Let $S(k) \mid p$. Then $S(k) = 1$ or $S(k) = p$. We prove that if $S(k) = p$, then $k \leq p!$. Indeed, this follows from the definition (12), since $S(k) = \min\{m \geq 1 : k \mid m!\} = p$ implies $k \mid p!$, so $k \leq p!$. Therefore the greatest value of k is $k = p!$, when $S(k) = p \mid p$. This proves relation (17).

Theorem 4. For all primes p,

$$S_{\min}(2p) \leq p^2 \leq S_{\max}(2p) \tag{18}$$

and more generally; for all $m \leq p$,

$$S_{\min}(mp) \leq p^m \leq S_{\max}(mp) \tag{19}$$

Proof. (19) follows by the known relation $S(p^m) = mp$ if $m \leq p$ and the definition (11), (13). Particularly, for $m = 2$, (19) reduces to (18). For $m = p$, (19) gives

$$S_{\min}(p^2) \leq p^p \leq S_{\max}(p^2) \tag{20}$$

This case when m is also an arbitrary prime is given in.

Theorem 5. For all odd primes p and $q, p < q$ one has

$$S_{\min}(pq) \leq q^p \leq p^q \leq S_{\max}(pq) \tag{21}$$

(21) holds also when $p = 2$ and $q \geq 5$.

Proof. Since $S(q^p) = pq$ and $S(p^q) = qp$ for primes p and q, the extreme inequalities of (21) follow from the definition (11) and (13). For the inequality $q^p < p^q$ remark that this is equivalent to $f(p) > f(q)$, where $f(x) = \frac{\ln x}{x} (x \geq 3)$.

Since $f'(x) = \frac{1-\ln x}{x^2} = 0 \Leftrightarrow x = e$ immediately follows that f is strictly decreasing for $x \geq e = 2.71\cdot$ From the graph of this function, since $\frac{\ln 2}{2} = \frac{\ln 4}{4}$ we get that

$$\frac{\ln 2}{2} < \frac{\ln 3}{3},$$

but

$$\frac{\ln 2}{2} > \frac{\ln q}{q}$$

for $q \geq 5$. Therefore (21) holds when $p = 2$ and $q \geq 5$. Indeed, $f(q) \leq f(5) < f(4) = f(2)$.

Remark. For all primes p, q

$$S_{\min}(pq) \leq \min\{p^q, q^p\} \tag{22}$$

and

$$S_{\max}(pq) \geq \max\{p^q, q^p\}. \tag{23}$$

For $p = q$ this implies relation (21).

Proof. Since $S(q^p) = S(p^q) = pq$, one has

$$S_{\min}(pq) \leq p^q, S_{\min}(pq) \leq q^p, S_{\max}(pq) \leq p^q, S_{\max}(pq) \leq q^p$$

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