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σ-Coloring of the Monohedral Tiling

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Abstract: In this paper we introduce the definition of σ-coloring and perfect σ-coloring for the plane which is equipped by tiling $\Im$. And we investigate the σ-coloring for the r-monohedral tiling.

Key Words: Smarandache k-tiling, coloring, Monohedral tiling

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§1. Introduction

For an integer $k \geq 1$, a Smarandache k-tiling of the plane is a family of sets called k-tiles covering each point in the plane exactly k times. Particularly, a Smarandache 1-tiling is usually called tiling of the plane [8]. Tilings are known as tessellations or pavings, they have appeared in human activities since prehistoric times. Their mathematical theory is mostly elementary, but nevertheless it contains a rich supply of interesting problems at various levels. The same is true for the special class of tiling called tiling by regular polygons [2]. The notions of tiling by regular polygons in the plane is introduced by Grunbaum and Shephard in [3]. For more details see [4, 5, 6, 7].

Definition 1.1 *A tiling of the plane is a collection $\Im = \{T_i : i \in I = \{1, 2, 3, ...\}\}$ of closed topological discs (tiles) which covers the Euclidean plane E^2 and is such that the interiors of the tiles are disjoint.*

More explicitly, the union of the sets $T_1, T_2, ...$, tiles, is to be the whole plane, and the interiors of the sets T_i topological disc it is meant a set whose boundary is a single simple closed curve. Two tiles are called adjacent if they have an edge in common, and then each is called an adjacent of the other. Two distinct edges are adjacent if they have a common endpoint. The word incident is used to denote the relation of a tile to each of its edges or vertices, and also of an edge to each of its endpoints. Two tilings $\Im_1$ and $\Im_2$ are congruent if $\Im_1$ may be made to coincide with $\Im_2$ by a rigid motion of the plane, possibly including reflection [1].

Definition 1.2 *A tiling is called edge-to-edge if the relation of any two tiles is one of the following three possibilities:*
(a) they are disjoint, or

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(b) they have precisely one common point which is a vertex of each the of polygons, or
(c) they share a segment that is an edge of each of the two polygons.

Hence a point of the plane that is a vertex of one of the polygons in an edge-to-edge tiling is also a vertex of every other polygon to which it belongs and it is called a vertex of the tiling. Similarly, each edge of one of the polygons, regular tiling, is an edge of precisely one other polygon and it is called an edge of the tiling. It should be noted that the only possible edge-to-edge tilings of the plane by mutually congruent regular convex polygons are the three regular tilings by equilateral triangles, by squares, or by regular hexagons. A portion of each of these three tilings is illustrated in Fig.1.

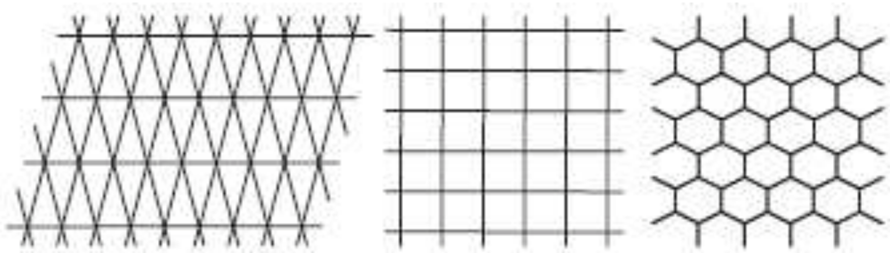

Fig.1

Definition 1.3 *The regular tiling $\Im$ will be called r-monohedral if every tile in $\Im$ is congruent to one fixed set T. The set T is called the prototile of $\Im$, where r is the number of vertices for each tile [2].*

§2. σ-Coloring

Let R^2 be equipped by r-monohedral tiling $\Im$, and let $V(\Im)$ be the set of all vertices of the tiling $\Im$.

Definition 2.1 *σ-coloring of the tiling $\Im$. Is a portion of $V(\Im)$ into k color classes such that:*

(i) different colors appears on adjacent vertices, and for each tile $T_i \in \Im$ there exist permutation σ from some color k.

(ii) The exist at least σ_i such that $O(\sigma_i) = k$. where $O(\sigma_i)$ is the order the permutation σ.

We will denote to the set of all permutation the σ-coloring by $J(\Im)$.

Definition 2.2 *The σ-coloring is called perfect σ-coloring if all tiles have the same permutation.*

Theorem 2.1 *The 3-monohedral tiling admit σ-coloring if and only if $k = 3$.*

Proof Let R^2 be equipped by 3-monohedral tiling $\Im$. If $k < 3$, then there exist adjacent vertices colored by the same color, which it contradicts with condition (i). If $k > 3$, then the condition (i) satisfied but the condition (ii) not satisfied because as we know that each tile has three vertices, so it cannot be colored by more than three colors, see Fig. 2.

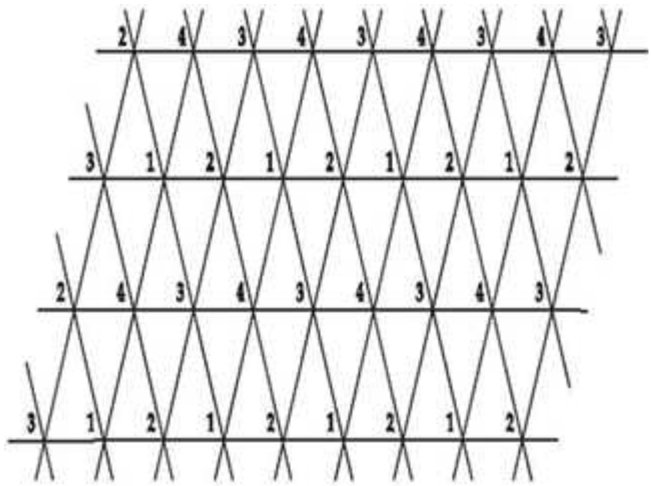

Fig.2

Hence the 3-monohedral tiling admit σ-coloring only if $k = 3$, and σwill be $\sigma = (123)$. see Fig.3.

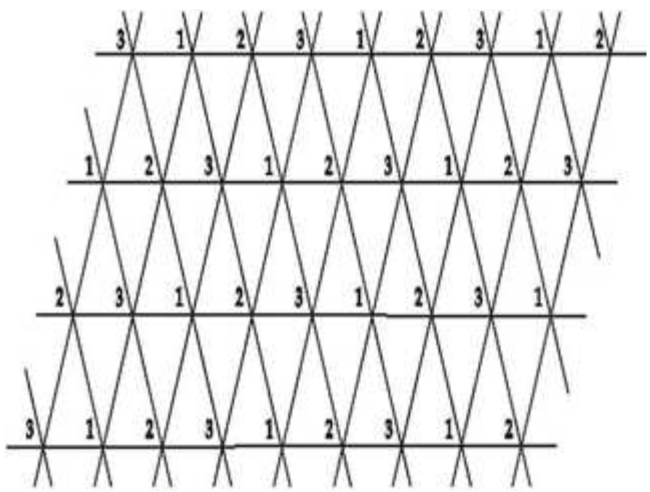

Fig.3

Corollary 2.1 *Every σ-coloring of 3-monohedral tiling is perfect σ-coloring.*

Theorem 2.2 *The 4-monohedral tiling admit σ-coloring if and only if $k = 2$ and $k = 4$.*

Proof Let R^2 be equipped by 4-monohedral tiling $\Im$. If the $k > 4$, then the condition (ii) not satisfied as in the 3-monohedral case, then k have only three cases $k = 1, 2, 3$ and 4. If $k = 1$then there exist adjacent vertices colored by the same color which contradicts this will contradicts with condition (i). If $k = 3$, the first three vertices colored by three colors, so the forth vertex colored by color differ from the color in the adjacent vertices by this way the tiling will be colored but this coloring is not σ-coloring since the permutation from colors not found.

Then the condition (i) not satisfied. If $k = 2$, the two condition of the σ-coloring are satisfied and so the tiling admits σ-coloring by permutation $\sigma = (12)$, for all tiles $T_i \in \Im$, see Fig. 4.

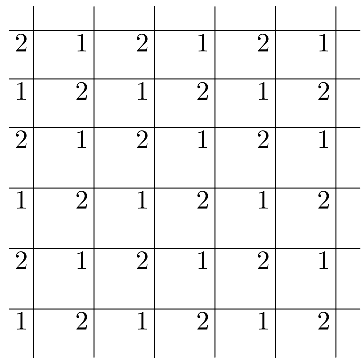

Fig.4

If $k = 4$, then $V(\Im)$ colored by four colors, and in this case the 5-monohedral tiling admit σ-coloring by two methods. The first method that all tiles have the permutation $\sigma = (1234)$, and in this case the σ-coloring will be perfect σ-coloring, see Fig. 5.

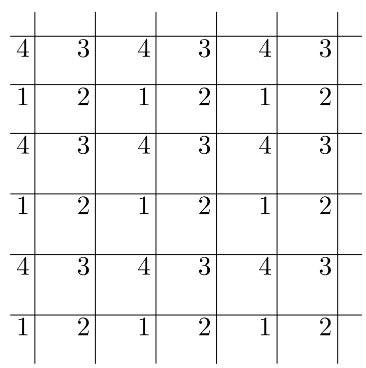

Fig.5

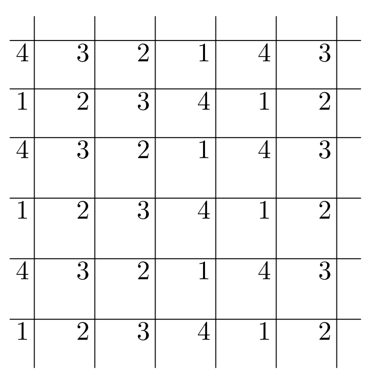

Fig.6

The second method, if the T_i has the permutation $\sigma = (1234)$. As we know, each tile surrounding by eight tiles four tiles adjacent to T_i by edges, and other four tiles adjacent to T_i, by vertices then we can colored these tiles by the four colors such that all tiles have the permutation $\sigma = (1234)$ and the tiles which adjacent by vertices will colored by some colors of k , such that each tile has one of these permutation $\{\alpha, \beta, \gamma, \delta\}$, where $\alpha = (12), \beta = (23), \gamma = (34)$ and $\delta = (41)$, see Figure 6. Then will be $J(\Im) = \{\sigma = (1234), \alpha = (12), \beta = (23), \gamma = (34), \delta = (41)\}$. □

Corollary 2.2 *The 4-monohedral tiling admit prefect σ-coloring if and only if $k = 2$.*

Theorem 2.3 *The 6-monohedral tiling admit σ-coloring if and only if $k = 2, k = 3$. and $k = 6$.*

Proof If $k > 6$, then the condition (ii) not satisfied as in the 3-monohedral case. Now we will investigate the cases where $k = 1, 2, 3, 4, 5$ and 6. If $k = 1$, then the condition (i) not satisfied as in the 4-monohedral case. If $k = 5$or $k = 4$ we known that each tile has six vertices, then in each case $k = 5$ or $k = 4$ the tiling can be colored by 5 or 4 colors but these colors not satisfied the condition (i). Then at $k = 5$or $k = 4$the coloring not be σ-coloring. If $k = 2$, the tiling coloring by two color then the vertices of each tile colored by two color, and the permutation will be $\sigma = (12)$ for all tiles see Fig. 7.

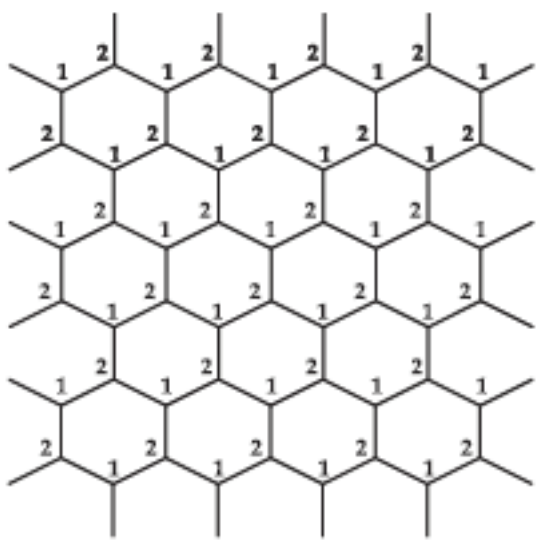

Fig.7

If $k = 3$, the tiling coloring by three colors. So suppose that T_i tile colored by $\sigma = (123)$, we know that each tile surrounding by six tiles, then if the tile which lies on edge $e_1 = (v_1 v_2) \in T_i$ has the permutation $\sigma = (123)$ thus the tiles which lies on $e_4 = (v_4 v_5)$ will colored by $\alpha = (12)$ and the converse is true. Similarly the edges $\{e_2, e_5\}$with the with the permutation $\{\sigma = (123456), \beta = (34)\}$ and $\{e_3, e_6\}$ with $\{\sigma = (123456), \delta = (56)\}$. Then permutation $\{\sigma = (123), \beta = (23)\}$ and $\{e_3, e_6\}$ with $\{\sigma = (123), \delta = (13)\}$, then $J(\Im) = \{\sigma = (123), \alpha = (12), \beta = (23), \delta = (13)\}$. see Fig. 8.

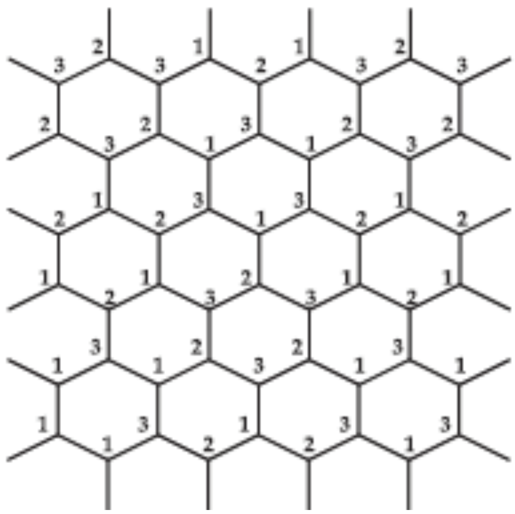

Fig.8

If $k = 6$. the tiling coloring by six colors. Then if the tile which lies on edge $e_1 = (v_1 v_2) \in T_i$ has the permutation $\sigma = (123456)$ thus the tiles which lies on $e_4 = (v_4 v_5)$ will colored by $\alpha = (12)$ and the converse is true. Similarly the edges $\{e_2, e_5\}$ $J(\Im) = \{\sigma = (123456), \alpha = (12), \beta = (34), \delta = (56)\}$, see Fig. 9. □

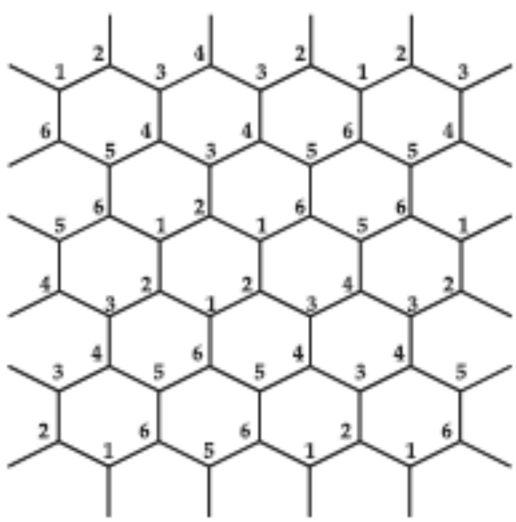

Fig.9

Corollary 2.3 *The 6 -monohedral tiling admit prefect σ-coloring if and only if $k = 2$.*

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