

Article

On Novel Results about the Algebraic Properties of Symbolic 3-Plithogenic and 4-Plithogenic Real Square Matrices

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Abstract: Symbolic n -plithogenic sets are considered to be modern concepts that carry within their framework both an algebraic and logical structure. The concept of symbolic n -plithogenic algebraic rings is considered to be a novel generalization of classical algebraic rings with many symmetric properties. These structures can be written as linear combinations of many symmetric elements taken from other classical algebraic structures, where the square symbolic k -plithogenic real matrices are square matrices with real symbolic k -plithogenic entries. In this research, we will find easy-to-use algorithms for calculating the determinant of a symbolic 3-plithogenic/4-plithogenic matrix, and for finding its inverse based on its classical components, and even for diagonalizing matrices of these types. On the other hand, we will present a new algorithm for calculating the eigenvalues and eigenvectors associated with matrices of these types. Also, the exponent of symbolic 3-plithogenic and 4-plithogenic real matrices will be presented, with many examples to clarify the novelty of this work.

Keywords: symbolic 3-plithogenic rings; symbolic 3-plithogenic real matrices; symbolic 4-plithogenic rings; symbolic 4-plithogenic real matrices; the diagonalization problem



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1. Introduction

Symbolic n -plithogenic sets were propounded by Smarandache in [1–3] to be a novel generalization of classical sets. These sets play a key role in the generalization of classical algebraic structures to new versions possessing several characteristic properties; in these we can see many symbolic n -plithogenic algebraic structures, such as symbolic 2-plithogenic structures [4–8], and symbolic 3-plithogenic algebraic structures [9–11]. For example, symbolic 2-plithogenic rings were used to generalize vector spaces and modules into symbolic 2-plithogenic spaces and modules. The same work was presented for symbolic 3-plithogenic rings [12,13]. Symbolic 2-plithogenic matrices were defined and studied in [14]; these matrices consist of symbolic 2-plithogenic real entries. These matrices are recognized as a similar structure of refined neutrosophic matrices and structures [15–24]. In matrix theory, it is very important to deal with the exponents of matrices and their related problems, such as how to diagonalize a matrix, and how to compute eigenvalues and eigenvectors.

From this point of view, we continue the previous efforts presented in order to further our understanding of the behavior of symbolic n -plithogenic matrices, and present some algorithms for computing inverses, eigen values and vectors, and the diagonalization of square matrices using symbolic 3-plithogenic entries; we also present some algorithms for computing inverses, eigen values and vectors, and the diagonalization of square matrices using symbolic 4-plithogenic entries.

Our study opens the door to many future applications of this type of matrices, especially those related to the problem of representation through linear transformations, or the representation of algebraic groups by matrices of this type.

2. Preliminaries

Definition 1. The ring of symbolic of 3-plithogenic real numbers is defined as follows:

$$3 - SP_R = \{x + yP_1 + zP_2 + tP_3; P_i \times P_j = P_{\max(i,j)}, P_i^2 = P_i\}$$

Addition on 3-SP_R is defined as follows:

$$[x_0 + x_1P_1 + x_2P_2 + x_3P_3] + [y_0 + y_1P_1 + y_2P_2 + y_3P_3] = (x_0 + y_0) + (x_1 + y_1)P_1 + (x_2 + y_2)P_2 + (x_3 + y_3)P_3$$

Multiplication on 3 - SP_R is defined as follows:

$$[x_0 + x_1P_1 + x_2P_2 + x_3P_3][y_0 + y_1P_1 + y_2P_2 + y_3P_3] = x_0y_0 + (x_0y_1 + x_1y_0 + x_1y_1)P_1 + (x_0y_2 + x_2y_1 + x_2y_2 + x_2y_0 + x_1y_2)P_2 + (x_0y_3 + x_1y_3 + x_2y_3 + x_3y_3 + x_3y_0 + x_3y_1 + x_3y_2)P_3$$

Remark 1. If we Let $X = x_0 + x_1P_1 + x_2P_2 + x_3P_3 \in 3 - SP_R$, we have the following:

X is invertible if, and only if, $x_0 \neq 0, x_0 + x_1 \neq 0, x_0 + x_1 + x_2 \neq 0, x_0 + x_1 + x_2 + x_3 \neq 0$, and $X^{-1} = \frac{1}{X} = \frac{1}{x_0} + \left[\frac{1}{x_0+x_1} - \frac{1}{x_0}\right]P_1 + \left[\frac{1}{x_0+x_1+x_2} - \frac{1}{x_0+x_1}\right]P_2 + \left[\frac{1}{x_0+x_1+x_2+x_3} - \frac{1}{x_0+x_1+x_2}\right]P_3$

For $n \in \mathbb{N}$, $X^n = x_0^n + [(x_0 + x_1)^n - x_0^n]P_1 + [(x_0 + x_1 + x_2)^n - (x_0 + x_1)^n]P_2 + [(x_0 + x_1 + x_2 + x_3)^n - (x_0 + x_1 + x_2)^n]P_3$.

Definition 2. The ring of symbolic of 4-plithogenic real numbers is defined as follows:

$$4 - SP_R = \{x + yP_1 + zP_2 + tP_3 + lP_4; P_i \times P_j = P_{\max(i,j)}, P_i^2 = P_i\}$$

Addition on 4 - SP_R is defined as follows:

$$[x_0 + x_1P_1 + x_2P_2 + x_3P_3 + x_4P_4] + [y_0 + y_1P_1 + y_2P_2 + y_3P_3 + y_4P_4] = (x_0 + y_0) + (x_1 + y_1)P_1 + (x_2 + y_2)P_2 + (x_3 + y_3)P_3 + (x_4 + y_4)P_4$$

Multiplication on 4-SP_R is defined as follows:

$$[x_0 + x_1P_1 + x_2P_2 + x_3P_3 + x_4P_4][y_0 + y_1P_1 + y_2P_2 + y_3P_3 + y_4P_4] = x_0y_0 + (x_0y_1 + x_1y_0 + x_1y_1)P_1 + (x_0y_2 + x_2y_1 + x_2y_2 + x_2y_0 + x_1y_2)P_2 + (x_0y_3 + x_1y_3 + x_2y_3 + x_3y_3 + x_3y_0 + x_3y_1 + x_3y_2)P_3 + (x_0y_4 + x_1y_4 + x_2y_4 + x_3y_4 + x_3y_4 + x_4y_0 + x_4y_1 + x_4y_2 + x_4y_3 + x_4y_4)P_4$$

3. Main Discussion

3.1. Symbolic 3-Plithogenic Matrices

Definition 3. A square symbolic 3-plithogenic matrix is a matrix with symbolic 3-plithogenic entries.

Example 1.

$$A = \begin{pmatrix} 4 + P_1 + P_2 - 2P_3 & 1 + P_3 & 3 - P_1 + 5P_3 \\ 5 - P_1 + P_2 & 2 - P_1 & 7P_1 + 4P_2 - P_3 \\ 6P_2 + 4P_3 & 5 + 11P_2 + P_3 & P_1 + P_3 \end{pmatrix}$$

is a symbolic 3-plithogenic 3 × 3 real matrix.

Remark 2. Any symbolic 3-plithogenic matrix can be written as follows: $A = A_0 + A_1P_1 + A_2P_2 + A_3P_3$.

For example, the matrix defined in the previous example can be written as follows:

$$A = \begin{pmatrix} 4 & 1 & 3 \\ 5 & 2 & 0 \\ 0 & 5 & 0 \end{pmatrix} + \begin{pmatrix} 1 & 0 & -1 \\ -1 & -1 & 7 \\ 0 & 0 & 1 \end{pmatrix} P_1 + \begin{pmatrix} 1 & 0 & 0 \\ 1 & 0 & 4 \\ 6 & 11 & 0 \end{pmatrix} P_2 + \begin{pmatrix} -2 & 1 & 5 \\ 0 & 0 & -1 \\ 4 & 1 & 1 \end{pmatrix} P_3$$

We denote the set of all symbolic 3-plithogenic n -square matrices by $(3 - SP_M)$. It is clear that $(3 - SP_M, +, \cdot)$ is a ring.

Theorem 1. Let $S = S_0 + S_1P_1 + S_2P_2 + S_3P_3 \in 3 - SP_M$, then the following is true:

1. S is invertible if, and only if, $S_0, S_0 + S_1, S_0 + S_1 + S_2, S_0 + S_1 + S_2 + S_3$ are invertible.
2. $S^{-1} = S_0^{-1} + [(S_0 + S_1)^{-1} - S_0^{-1}]P_1 + [(S_0 + S_1 + S_2)^{-1} - (S_0 + S_1)^{-1}]P_2 + [(S_0 + S_1 + S_2 + S_3)^{-1} - (S_0 + S_1 + S_2)^{-1}]P_3$
3. For $n \in \mathbb{N}$, $S^n = S_0^n + [(S_0 + S_1)^n - S_0^n]P_1 + [(S_0 + S_1 + S_2)^n - (S_0 + S_1)^n]P_2 + [(S_0 + S_1 + S_2 + S_3)^n - (S_0 + S_1 + S_2)^n]P_3$
4. $e^S = e^{S_0} + [e^{S_0+S_1} - e^{S_0}]P_1 + [e^{S_0+S_1+S_2} - e^{S_0+S_1}]P_2 + [e^{S_0+S_1+S_2+S_3} - e^{S_0+S_1+S_2}]P_3$.

Proof of Theorem 1. (1,2). Assume that $S_0, S_0 + S_1, S_0 + S_1 + S_2, S_0 + S_1 + S_2 + S_3$ are invertible. We have the following:

$$K = S_0^{-1} + [(S_0 + S_1)^{-1} - S_0^{-1}]P_1 + [(S_0 + S_1 + S_2)^{-1} - (S_0 + S_1)^{-1}]P_2 + [(S_0 + S_1 + S_2 + S_3)^{-1} - (S_0 + S_1 + S_2)^{-1}]P_3$$

Let us compute the following:

$$\begin{aligned} SK &= S_0S_0^{-1} + [S_0(S_0 + S_1)^{-1} - S_0S_0^{-1} + S_1S_0^{-1} + S_1(S_0 + S_1)^{-1} - S_1S_0^{-1}]P_1 \\ &\quad + [S_0(S_0 + S_1 + S_2)^{-1} - S_0(S_0 + S_1)^{-1} + S_1(S_0 + S_1 + S_2)^{-1} \\ &\quad - S_1(S_0 + S_1)^{-1} + S_2(S_0 + S_1 + S_2)^{-1} - S_2(S_0 + S_1)^{-1} + S_2S_0^{-1} \\ &\quad + S_2(S_0 + S_1)^{-1} - S_2S_0^{-1}]P_2 + \\ &\quad [S_0(S_0 + S_1 + S_2 + S_3)^{-1} - S_0(S_0 + S_1 + S_2)^{-1} + S_1(S_0 + S_1 + S_2 + S_3)^{-1} \\ &\quad - S_1(S_0 + S_1 + S_2)^{-1} + S_2(S_0 + S_1 + S_2 + S_3)^{-1} - S_2(S_0 + S_1 + S_2)^{-1} \\ &\quad + S_3(S_0 + S_1 + S_2 + S_3)^{-1} - S_3(S_0 + S_1 + S_2)^{-1} + S_3S_0^{-1} + S_3(S_0 + S_1)^{-1} \\ &\quad - S_3S_0^{-1} + S_3(S_0 + S_1 + S_2 + S_3)^{-1} - S_3(S_0 + S_1)^{-1}]P_3 = \\ U_{n \times n} &+ [O_{n \times n}]P_1 + [O_{n \times n}]P_2 + [O_{n \times n}]P_3 = U_{n \times n} \end{aligned}$$

Thus, S is invertible and $S^{-1} = K$.

For the converse, if we suppose that S is invertible, then there exists $K = K_0 + K_1P_1 + K_2P_2 + K_3P_3 \in 3 - SP_M$ such that $S \times K = U_{n \times n}$. This is equivalent to the following:

$$\begin{cases} S_0K_0 = U_{n \times n} & (1) \\ S_0K_1 + S_1K_0 + S_1K_1 = O_{n \times n} & (2) \\ S_0K_2 + S_1K_2 + S_2K_2 + S_2K_1 + S_2K_0 = O_{n \times n} & (3) \\ S_0K_3 + S_1K_3 + S_2K_3 + S_3K_3 + S_3K_0 + S_3K_1 + S_3K_2 = O_{n \times n} & (4) \end{cases}$$

Equation (1) implies that S_0 is invertible and $S_0^{-1} = K_0$.

By adding (1) to (2), we obtain $(S_0 + S_1)(K_0 + K_1) = U_{n \times n}$, so that $S_0 + S_1$ is invertible and $(S_0 + S_1)^{-1} = K_0 + K_1$, hence $K_1 = (S_0 + S_1)^{-1} - S_0^{-1}$.

By adding (1) to (2) to (3), we obtain $(S_0 + S_1 + S_2)(K_0 + K_1 + K_2) = U_{n \times n}$, so that $S_0 + S_1 + S_2$ is invertible and $(S_0 + S_1 + S_2)^{-1} = K_0 + K_1 + K_2$, hence $K_2 = (S_0 + S_1 + S_2)^{-1} - (S_0 + S_1)^{-1}$.

By adding (1) to (2) to (3) to (4), we obtain $(S_0 + S_1 + S_2 + S_3)(K_0 + K_1 + K_2 + K_3) = U_{n \times n}$, so that $S_0 + S_1 + S_2 + S_3$ is invertible and $(S_0 + S_1 + S_2 + S_3)^{-1} = K_0 + K_1 + K_2 + K_3$, hence $K_3 = (S_0 + S_1 + S_2 + S_3)^{-1} - (S_0 + S_1 + S_2)^{-1}$.

(3). For $n = 1$, it holds directly. We suppose that it is true. For $n = k$, we must prove it for $k + 1$.

$$\begin{aligned}
 S^{k+1} = SS^k = S_0S_0^k &+ \left[S_0(S_0 + S_1)^k - S_0S_0^k + S_1S_0^k + S_1(S_0 + S_1)^k - S_1S_0^k \right] P_1 \\
 &+ \left[S_0(S_0 + S_1 + S_2)^k - S_0(S_0 + S_1)^k + S_1(S_0 + S_1 + S_2)^k - S_1(S_0 + S_1)^k + S_2(S_0 + S_1 + S_2)^k \right. \\
 &\quad \left. - S_2(S_0 + S_1)^k + S_2S_0^k + S_2(S_0 + S_1)^k - S_2S_0^k \right] P_2 \\
 &+ \left[S_0(S_0 + S_1 + S_2 + S_3)^k - S_0(S_0 + S_1 + S_2)^k + S_1(S_0 + S_1 + S_2 + S_3)^k - S_1(S_0 + S_1 + S_2)^k \right. \\
 &\quad \left. + S_2(S_0 + S_1 + S_2 + S_3)^k - S_2(S_0 + S_1 + S_2)^k + S_3(S_0 + S_1 + S_2 + S_3)^k - S_3(S_0 + S_1 + S_2)^k \right. \\
 &\quad \left. + S_3S_0^k + S_3(S_0 + S_1)^k - S_3S_0^k + S_3(S_0 + S_1 + S_2 + S_3)^k - S_3(S_0 + S_1)^k \right] P_3 \\
 &= S_0^{k+1} + \left[(S_0 + S_1)(S_0 + S_1)^k - S_0S_0^k \right] P_1 \\
 &+ \left[(S_0 + S_1 + S_2)(S_0 + S_1 + S_2)^k - (S_0 + S_1)(S_0 + S_1)^k \right] P_2 \\
 &+ \left[(S_0 + S_1 + S_2 + S_3)(S_0 + S_1 + S_2 + S_3)^k - (S_0 + S_1 + S_2)(S_0 + S_1 + S_2)^k \right] P_3 \\
 &= S_0^{k+1} + \left[(S_0 + S_1)^{k+1} - S_0S_0^{k+1} \right] P_1 + \left[(S_0 + S_1 + S_2)^{k+1} - (S_0 + S_1)^{k+1} \right] P_2 \\
 &+ \left[(S_0 + S_1 + S_2 + S_3)^{k+1} - (S_0 + S_1 + S_2)^{k+1} \right] P_3
 \end{aligned}$$

Therefore, the proof holds via induction.

$$\begin{aligned}
 (4). e^S = \sum_{n=0}^{\infty} \frac{S^n}{n!} &= \sum_{n=0}^{\infty} \frac{S_0^n}{n!} + \left[\sum_{n=0}^{\infty} \frac{(S_0+S_1)^n}{n!} - \sum_{n=0}^{\infty} \frac{S_0^n}{n!} \right] P_1 + \left[\sum_{n=0}^{\infty} \frac{(S_0+S_1+S_2)^n}{n!} - \sum_{n=0}^{\infty} \frac{(S_0+S_1)^n}{n!} \right] P_2 \\
 &+ \left[\sum_{n=0}^{\infty} \frac{(S_0+S_1+S_2+S_3)^n}{n!} - \sum_{n=0}^{\infty} \frac{(S_0+S_1+S_2)^n}{n!} \right] P_3 = e^{S_0} + [e^{S_0+S_1} - e^{S_0}] P_1 + [e^{S_0+S_1+S_2} - e^{S_0+S_1}] P_2 \\
 &+ [e^{S_0+S_1+S_2+S_3} - e^{S_0+S_1+S_2}] P_3. \quad \square
 \end{aligned}$$

Definition 4. Let $S = S_0 + S_1P_1 + S_2P_2 + S_3P_3 \in 3 - SP_M$, then we define the following:

$$\det(S) = \det(S_0) + [\det(S_0 + S_1) - \det(S_0)]P_1 + [\det(S_0 + S_1 + S_2) - \det(S_0 + S_1)]P_2 + [\det(S_0 + S_1 + S_2 + S_3) - \det(S_0 + S_1 + S_2)]P_3$$

Theorem 2. S is invertible if, and only if, $\det(S)$ is invertible in $3 - SP_R$.

The proof holds via a similar discussion of the 2-plithogenic case.

Example 2. Consider the following matrix:

$$\begin{aligned}
 S &= \begin{pmatrix} 1 + P_1 + P_2 + P_3 & 5 + P_1 - 2P_2 \\ 1 + 4P_1 - P_2 + 3P_3 & P_1 + P_3 \end{pmatrix} \\
 &= \begin{pmatrix} 1 & 5 \\ 1 & 0 \end{pmatrix} + \begin{pmatrix} 1 & 1 \\ 4 & 1 \end{pmatrix} P_1 + \begin{pmatrix} 1 & -2 \\ -1 & 0 \end{pmatrix} P_2 + \begin{pmatrix} 1 & 0 \\ 3 & 1 \end{pmatrix} P_3
 \end{aligned}$$

We have $\det(S_0) = -5$, $\det(S_0 + S_1) = \det \begin{pmatrix} 2 & 6 \\ 5 & 1 \end{pmatrix} = -28$, $\det(S_0 + S_1 + S_2) = \det \begin{pmatrix} 3 & 4 \\ 4 & 1 \end{pmatrix} = -13$, $\det(S_0 + S_1 + S_2 + S_3) = \det \begin{pmatrix} 4 & 4 \\ 7 & 2 \end{pmatrix} = -20$.
Hence,

$$\det(S) = -5 + [-28 + 5]P_1 + [-13 + 28]P_2 + [-20 + 13]P_3 = -5 - 23P_1 + 15P_2 - 7P_3$$

Since $\det(S)$ is invertible, then S is invertible.

$$\begin{aligned}
 S_0^{-1} &= \frac{-1}{5} \begin{pmatrix} 0 & -5 \\ -1 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ \frac{1}{5} & \frac{-1}{5} \end{pmatrix} \\
 (S_0 + S_1)^{-1} &= \frac{-1}{28} \begin{pmatrix} 1 & -6 \\ -5 & 2 \end{pmatrix} = \begin{pmatrix} \frac{-1}{28} & \frac{6}{28} \\ \frac{5}{28} & \frac{-2}{28} \end{pmatrix} \\
 (S_0 + S_1 + S_2)^{-1} &= \frac{-1}{13} \begin{pmatrix} 1 & -4 \\ -4 & 3 \end{pmatrix} = \begin{pmatrix} \frac{-1}{13} & \frac{4}{13} \\ \frac{4}{13} & \frac{-3}{13} \end{pmatrix} \\
 (S_0 + S_1 + S_2 + S_3)^{-1} &= \frac{-1}{20} \begin{pmatrix} 2 & -4 \\ -7 & 4 \end{pmatrix} = \begin{pmatrix} \frac{-2}{20} & \frac{4}{20} \\ \frac{7}{20} & \frac{-4}{20} \end{pmatrix} \\
 S^{-1} &= S_0^{-1} + \left[(S_0 + S_1)^{-1} - S_0^{-1} \right] P_1 + \left[(S_0 + S_1 + S_2)^{-1} - (S_0 + S_1)^{-1} \right] P_2 + \\
 &\quad \left[(S_0 + S_1 + S_2 + S_3)^{-1} - (S_0 + S_1 + S_2)^{-1} \right] P_3 \\
 S^{-1} &= \begin{pmatrix} 0 & 1 \\ \frac{1}{5} & \frac{-1}{5} \end{pmatrix} + \left[\begin{pmatrix} \frac{-1}{28} & \frac{-22}{28} \\ \frac{-3}{28} & \frac{18}{28} \end{pmatrix} \right] P_1 + \left[\begin{pmatrix} \frac{-15}{364} & \frac{34}{364} \\ \frac{47}{364} & \frac{-58}{364} \end{pmatrix} \right] P_2 + \left[\begin{pmatrix} \frac{-6}{260} & \frac{-28}{260} \\ \frac{11}{260} & \frac{8}{260} \end{pmatrix} \right] P_3 = \\
 &\quad \begin{pmatrix} -\frac{1}{28}P_1 - \frac{15}{364}P_2 - \frac{6}{260}P_3 & 1 - \frac{22}{28}P_1 + \frac{34}{364}P_2 - \frac{28}{260}P_3 \\ \frac{1}{5} - \frac{3}{28}P_1 + \frac{47}{364}P_2 + \frac{11}{260}P_3 & -\frac{1}{5} + \frac{18}{28}P_1 + \frac{58}{364}P_2 + \frac{8}{260}P_3 \end{pmatrix}
 \end{aligned}$$

3.2. Symbolic 3-Plithogenic Eigen Values/Vectors

Theorem 3. Let $S = S_0 + S_1P_1 + S_2P_2 + S_3P_3 \in 3 - SP_M$, then $A = a_0 + a_1P_1 + a_2P_2 + a_3P_3$ is an eigen value of S if, and only if, a_0 eigen value of S_0 , $a_0 + a_1$ eigen value of $S_0 + S_1$, $a_0 + a_1 + a_2$ eigen value of $S_0 + S_1 + S_2$, $a_0 + a_1 + a_2 + a_3$ eigen value of $S_0 + S_1 + S_2 + S_3$.

In addition, $X = X_0 + X_1P_1 + X_2P_2 + X_3P_3$ is the corresponding eigen vector of A if, and only if, X_0 eigen vector of a_0 , $X_0 + X_1$ eigen vector of $a_0 + a_1$, $X_0 + X_1 + X_2$ eigen vector of $a_0 + a_1 + a_2$, $X_0 + X_1 + X_2 + X_3$ eigen vector of $a_0 + a_1 + a_2 + a_3$.

Proof of Theorem 3. From the equation $AS = AX$, we can obtain:

$$\begin{cases}
 a_0S_0 = a_0X_0 & (5) \\
 a_0S_1 + a_1S_0 + a_1S_1 = a_0X_1 + a_1X_0 + a_1X_1 & (6) \\
 a_0S_2 + a_1S_2 + a_2S_2 + a_2S_1 + a_2S_0 = a_0X_2 + a_1X_2 + a_2X_2 + a_2X_1 + a_2X_0 & (7) \\
 a_0S_3 + a_1S_3 + a_2S_3 + a_3S_3 + a_3S_0 + a_3S_1 + a_3S_2 = a_0X_3 + a_1X_3 + a_2X_3 + a_3X_3 + a_3X_0 + a_3X_1 + a_3X_2 & (8)
 \end{cases}$$

Equation (5) implies that a_0 is the eigen value of S_0 , with X_0 as the eigen vector.

By adding (5) to (6), we obtain $(a_0 + a_1)(S_0 + S_1) = (a_0 + a_1)(X_0 + X_1)$, which means that $a_0 + a_1$ is the eigen value of $S_0 + S_1$, with $X_0 + X_1$ as the eigen vector.

By adding (5) to (6) to (7), we obtain $(a_0 + a_1 + a_2)(S_0 + S_1 + S_2) = (a_0 + a_1 + a_2)(X_0 + X_1 + X_2)$, which means that $a_0 + a_1 + a_2$ is the eigen value of $S_0 + S_1 + S_2$, with $X_0 + X_1 + X_2$ as the eigen vector.

By adding (5) to (6) to (7) to (8), we obtain $(a_0 + a_1 + a_2 + a_3)(S_0 + S_1 + S_2 + S_3) = (a_0 + a_1 + a_2 + a_3)(X_0 + X_1 + X_2 + X_3)$, which means that $a_0 + a_1 + a_2 + a_3$ is the eigen value of $S_0 + S_1 + S_2 + S_3$, with $X_0 + X_1 + X_2 + X_3$ as the eigen vector.

Therefore, the proof is complete. \square

Example 3. Consider the following matrix:

$$S = \begin{pmatrix} 3 + 2P_1 - P_3 & 4 + P_2 - 3P_3 \\ 2P_1 + P_2 + P_3 & 5 + P_1 \end{pmatrix} = \begin{pmatrix} 3 & 4 \\ 0 & 5 \end{pmatrix} + \begin{pmatrix} 2 & 0 \\ 2 & 1 \end{pmatrix} P_1 + \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} P_2 + \begin{pmatrix} -1 & -3 \\ 1 & 0 \end{pmatrix} P_3 = A_0 + A_1P_1 + A_2P_2 + A_3P_3$$

The eigen values of $A_0 = \begin{pmatrix} 3 & 4 \\ 0 & 5 \end{pmatrix}$ are $\{3, 5\}$.

The eigen values of $A_0 + A_1 = \begin{pmatrix} 2 & 0 \\ 2 & 1 \end{pmatrix}$ are $\left\{ \frac{11+\sqrt{33}}{2}, \frac{11-\sqrt{33}}{2} \right\}$.

The eigen values of $A_0 + A_1 + A_2 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ are $\left\{ \frac{11+\sqrt{61}}{2}, \frac{11-\sqrt{61}}{2} \right\}$.

The eigen values of $A_0 + A_1 + A_2 + A_3 = \begin{pmatrix} -1 & -3 \\ 1 & 0 \end{pmatrix}$ are $\{8, 2\}$.

The eigen values of the symbolic 3-plithogenic matrix A are as follows:

$$U_1 = 3 + \left(\frac{11 + \sqrt{33}}{2} - 3 \right) P_1 + \left(\frac{11 + \sqrt{61}}{2} - \frac{11 + \sqrt{33}}{2} \right) P_2 + \left(8 - \frac{11 + \sqrt{61}}{2} \right) P_3$$

$$U_2 = 3 + \left(\frac{11 + \sqrt{33}}{2} - 3 \right) P_1 + \left(\frac{11 + \sqrt{61}}{2} - \frac{11 + \sqrt{33}}{2} \right) P_2 + \left(2 - \frac{11 + \sqrt{61}}{2} \right) P_3$$

$$U_3 = 3 + \left(\frac{11 + \sqrt{33}}{2} - 3 \right) P_1 + \left(\frac{11 - \sqrt{61}}{2} - \frac{11 + \sqrt{33}}{2} \right) P_2 + \left(8 - \frac{11 + \sqrt{61}}{2} \right) P_3$$

$$U_4 = 3 + \left(\frac{11 + \sqrt{33}}{2} - 3 \right) P_1 + \left(\frac{11 - \sqrt{61}}{2} - \frac{11 + \sqrt{33}}{2} \right) P_2 + \left(2 - \frac{11 + \sqrt{61}}{2} \right) P_3$$

$$U_5 = 3 + \left(\frac{11 - \sqrt{33}}{2} - 3 \right) P_1 + \left(\frac{11 + \sqrt{61}}{2} - \frac{11 - \sqrt{33}}{2} \right) P_2 + \left(8 - \frac{11 + \sqrt{61}}{2} \right) P_3$$

$$U_6 = 3 + \left(\frac{11 - \sqrt{33}}{2} - 3 \right) P_1 + \left(\frac{11 - \sqrt{61}}{2} - \frac{11 - \sqrt{33}}{2} \right) P_2 + \left(2 - \frac{11 - \sqrt{61}}{2} \right) P_3$$

$$U_7 = 3 + \left(\frac{11 - \sqrt{33}}{2} - 3 \right) P_1 + \left(\frac{11 - \sqrt{61}}{2} - \frac{11 - \sqrt{33}}{2} \right) P_2 + \left(8 - \frac{11 - \sqrt{61}}{2} \right) P_3$$

$$U_8 = 3 + \left(\frac{11 - \sqrt{33}}{2} - 3 \right) P_1 + \left(\frac{11 + \sqrt{61}}{2} - \frac{11 - \sqrt{33}}{2} \right) P_2 + \left(2 - \frac{11 - \sqrt{61}}{2} \right) P_3$$

$$U_9 = 5 + \left(\frac{11 + \sqrt{33}}{2} - 5 \right) P_1 + \left(\frac{11 + \sqrt{61}}{2} - \frac{11 + \sqrt{33}}{2} \right) P_2 + \left(8 - \frac{11 + \sqrt{61}}{2} \right) P_3$$

$$U_{10} = 5 + \left(\frac{11 + \sqrt{33}}{2} - 5 \right) P_1 + \left(\frac{11 + \sqrt{61}}{2} - \frac{11 + \sqrt{33}}{2} \right) P_2 + \left(2 - \frac{11 + \sqrt{61}}{2} \right) P_3$$

$$U_{11} = 5 + \left(\frac{11 + \sqrt{33}}{2} - 5 \right) P_1 + \left(\frac{11 - \sqrt{61}}{2} - \frac{11 + \sqrt{33}}{2} \right) P_2 + \left(8 - \frac{11 - \sqrt{61}}{2} \right) P_3$$

$$U_{12} = 5 + \left(\frac{11 + \sqrt{33}}{2} - 5 \right) P_1 + \left(\frac{11 - \sqrt{61}}{2} - \frac{11 + \sqrt{33}}{2} \right) P_2 + \left(2 - \frac{11 - \sqrt{61}}{2} \right) P_3$$

$$U_{13} = 5 + \left(\frac{11 - \sqrt{33}}{2} - 5\right)P_1 + \left(\frac{11 + \sqrt{61}}{2} - \frac{11 - \sqrt{33}}{2}\right)P_2 + \left(8 - \frac{11 + \sqrt{61}}{2}\right)P_3$$

$$U_{14} = 5 + \left(\frac{11 - \sqrt{33}}{2} - 5\right)P_1 + \left(\frac{11 + \sqrt{61}}{2} - \frac{11 - \sqrt{33}}{2}\right)P_2 + \left(2 - \frac{11 + \sqrt{61}}{2}\right)P_3$$

$$U_{15} = 5 + \left(\frac{11 - \sqrt{33}}{2} - 5\right)P_1 + \left(\frac{11 - \sqrt{61}}{2} - \frac{11 - \sqrt{33}}{2}\right)P_2 + \left(8 - \frac{11 - \sqrt{61}}{2}\right)P_3$$

$$U_{16} = 5 + \left(\frac{11 - \sqrt{33}}{2} - 5\right)P_1 + \left(\frac{11 + \sqrt{61}}{2} - \frac{11 - \sqrt{33}}{2}\right)P_2 + \left(2 - \frac{11 - \sqrt{61}}{2}\right)P_3$$

Remark 3. If A has n eigen values for $0 \leq i \leq s$, then A has n^4 eigen values.

3.3. Symbolic 3-Plithogenic Diagonalization

Theorem 4. Let $T = T_0 + T_1P_1 + T_2P_2 + T_3P_3$ be an n -square symbolic 3-plithogenic matrix, then T is diagonalizable if, and only if, $T_0, T_0 + T_1, T_0 + T_1 + T_2, T_0 + T_1 + T_2 + T_3$ are diagonalizable.

Proof of Theorem 4. T are diagonalizable if, and only if, there exists $L = L_0 + L_1P_1 + L_2P_2 + L_3P_3$ and $D = D_0 + D_1P_1 + D_2P_2 + D_3P_3$ such that $T = L^{-1}DL$, where D is diagonal and L is invertible.

The equation $T = L^{-1}DL$ is equivalent to the following:

$$\begin{cases} T_0 = L_0^{-1}D_0L_0 \\ T_0 + T_1 = (L_0 + L_1)^{-1}(D_0 + D_1)(L_0 + L_1) \\ T_0 + T_1 + T_2 = (L_0 + L_1 + L_2)^{-1}(D_0 + D_1 + D_2)(L_0 + L_1 + L_2) \\ T_0 + T_1 + T_2 + T_3 = (L_0 + L_1 + L_2 + L_3)^{-1}(D_0 + D_1 + D_2 + D_3)(L_0 + L_1 + L_2 + L_3) \end{cases}$$

Which implies the proof. \square

Example 4. Consider the following matrix:

$$\begin{aligned} T &= \begin{pmatrix} 2 + P_1 + P_2 + 2P_3 & P_1 + P_2 \\ 3 + P_3 & -1 + 3P_2 + P_3 \end{pmatrix} \\ &= \begin{pmatrix} 2 & 0 \\ 3 & -1 \end{pmatrix} + \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}P_1 + \begin{pmatrix} 1 & 1 \\ 0 & 3 \end{pmatrix}P_2 + \begin{pmatrix} 2 & 0 \\ 1 & 1 \end{pmatrix}P_3 \\ &= T_0 + T_1P_1 + T_2P_2 + T_3P_3 \end{aligned}$$

$$T_0 = \begin{pmatrix} 2 & 0 \\ 3 & -1 \end{pmatrix} = L_0D_0L_0^{-1} = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} 2 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}^{-1}$$

$$\begin{aligned} T_0 + T_1 &= \begin{pmatrix} 3 & 1 \\ 3 & -1 \end{pmatrix} = M_1K_1M_1^{-1} \\ &= \begin{pmatrix} 1 & 1 \\ -2 + \sqrt{7} & -2 - \sqrt{7} \end{pmatrix} \begin{pmatrix} 1 + \sqrt{7} & 0 \\ 0 & 1 - \sqrt{7} \end{pmatrix} \begin{pmatrix} 1 & 1 \\ -2 + \sqrt{7} & -2 - \sqrt{7} \end{pmatrix}^{-1} \end{aligned}$$

$$\begin{aligned}
 T_0 + T_1 + T_2 &= \begin{pmatrix} 4 & 2 \\ 3 & 2 \end{pmatrix} = M_2 K_2 M_2^{-1} \\
 &= \begin{pmatrix} 1 & 1 \\ \frac{-1+\sqrt{7}}{2} & \frac{-1-\sqrt{7}}{2} \end{pmatrix} \begin{pmatrix} 3+\sqrt{7} & 0 \\ 0 & 3-\sqrt{7} \end{pmatrix} \begin{pmatrix} 1 & 1 \\ \frac{-1+\sqrt{7}}{2} & \frac{-1-\sqrt{7}}{2} \end{pmatrix}^{-1} \\
 T_0 + T_1 + T_2 + T_3 &= \begin{pmatrix} 6 & 2 \\ 4 & 3 \end{pmatrix} = M_3 K_3 M_3^{-1} \\
 &= \begin{pmatrix} 1 & 1 \\ \frac{-3+\sqrt{41}}{4} & \frac{-3-\sqrt{41}}{4} \end{pmatrix} \begin{pmatrix} \frac{9+\sqrt{41}}{4} & 0 \\ 0 & \frac{9-\sqrt{41}}{4} \end{pmatrix} \begin{pmatrix} 1 & 1 \\ \frac{-3+\sqrt{41}}{4} & \frac{-3-\sqrt{41}}{4} \end{pmatrix}^{-1}
 \end{aligned}$$

So that $L_1 = M_1 - L_0, L_2 = M_2 - M_1, L_3 = M_3 - M_2, D_1 = K_1 - D_0, D_2 = K_2 - K_1, D_3 = K_3 - K_2.$

3.4. Symbolic 4-Plithogenic Matrices

Definition 5. A square symbolic 4-plithogenic matrix is a matrix with symbolic 4-plithogenic entries.

Example 5.

$$A = \begin{pmatrix} 4 + P_1 + P_2 - 2P_3 + 4P_4 & 1 + P_3 + 2P_4 & 3 - P_1 + 5P_3 + 4P_4 \\ 5 - P_1 + P_2 + 3P_4 & 2 - P_1 + 11P_4 & 7P_1 + 4P_2 - P_3 + 6P_4 \\ 6P_2 + 4P_3 + 2P_4 & 5 + 11P_2 + P_3 + 11P_4 & P_1 + P_3 + 3P_4 \end{pmatrix}$$

is a symbolic 4-plithogenic 3×3 real matrix.

Remark 4. Any symbolic 4-plithogenic matrix can be written as follows: $A = A_0 + A_1P_1 + A_2P_2 + A_3P_3 + A_4P_4.$

For example, the matrix defined in the previous example can be written as follows:

$$A = \begin{pmatrix} 4 & 1 & 3 \\ 5 & 2 & 0 \\ 0 & 5 & 0 \end{pmatrix} + \begin{pmatrix} 1 & 0 & -1 \\ -1 & -1 & 7 \\ 0 & 0 & 1 \end{pmatrix} P_1 + \begin{pmatrix} 1 & 0 & 0 \\ 1 & 0 & 4 \\ 6 & 11 & 0 \end{pmatrix} P_2 + \begin{pmatrix} -2 & 1 & 5 \\ 0 & 0 & -1 \\ 4 & 1 & 1 \end{pmatrix} P_3 + \begin{pmatrix} 4 & 2 & 4 \\ 3 & 11 & 6 \\ 2 & 11 & 3 \end{pmatrix} P_4$$

Definition 6. Let $S = S_0 + S_1P_1 + S_2P_2 + S_3P_3 + S_4P_4, T = T_0 + T_1P_1 + T_2P_2 + T_3P_3 + T_4P_4$ be two n -square symbolic 4-plithogenic matrices, then the following is true:

$$S + T = (S_0 + T_0) + (S_1 + T_1)P_1 + (S_2 + T_2)P_2 + (S_3 + T_3)P_3 + (S_4 + T_4)P_4$$

$$\begin{aligned}
 S \times T &= S_0T_0 + (S_0T_1 + S_1T_0 + S_1T_1)P_1 + (S_0T_2 + S_2T_1 + S_2T_2 + S_2T_0 + S_1T_2)P_2 + (S_0T_3 + S_1T_3 + S_2T_3 + S_3T_3 + S_3T_0 \\
 &+ S_3T_1 + S_3T_2)P_3 + (S_0T_4 + S_1T_4 + S_2T_4 + S_3T_4 + S_4T_0 + S_4T_1 + S_4T_2 + S_4T_3 + S_4T_4)P_4
 \end{aligned}$$

We denote the set of all symbolic 4-plithogenic n -square matrices using $(4 - SP_M).$ It is clear that $(4 - SP_M, +, \cdot).$ Is a ring.

Theorem 5. Let $S = S_0 + S_1P_1 + S_2P_2 + S_3P_3 + S_4P_4 \in 4 - SP_M,$ then:

1. S is invertible if, and only if, $S_0, S_0 + S_1, S_0 + S_1 + S_2, S_0 + S_1 + S_2 + S_3, S_0 + S_1 + S_2 + S_3 + S_4$ are invertible.
2. $S^{-1} = S_0^{-1} + [(S_0 + S_1)^{-1} - S_0^{-1}]P_1 + [(S_0 + S_1 + S_2)^{-1} - (S_0 + S_1)^{-1}]P_2 + [(S_0 + S_1 + S_2 + S_3)^{-1} - (S_0 + S_1 + S_2)^{-1}]P_3 + [(S_0 + S_1 + S_2 + S_3 + S_4)^{-1} - (S_0 + S_1 + S_2 + S_3)^{-1}]P_4$
3. For $n \in \mathbb{N}$

4. $S^n = S_0^n + [(S_0 + S_1)^n - S_0^n]P_1 + [(S_0 + S_1 + S_2)^n - (S_0 + S_1)^n]P_2 + [(S_0 + S_1 + S_2 + S_3)^n - (S_0 + S_1 + S_2)^n]P_3 + [(S_0 + S_1 + S_2 + S_3 + S_4)^n - (S_0 + S_1 + S_2 + S_3)^n]P_4.$
5. $e^S = e^{S_0} + [e^{S_0+S_1} - e^{S_0}]P_1 + [e^{S_0+S_1+S_2} - e^{S_0+S_1}]P_2 + [e^{S_0+S_1+S_2+S_3} - e^{S_0+S_1+S_2}]P_3 + [e^{S_0+S_1+S_2+S_3+S_4} - e^{S_0+S_1+S_2+S_3}]P_4$

Proof of Theorem 5. (1,2). Assume that $S_0, S_0 + S_1, S_0 + S_1 + S_2, S_0 + S_1 + S_2 + S_3, S_0 + S_1 + S_2 + S_3 + S_4$ are invertible. We have the following:

$$K = S_0^{-1} + [(S_0 + S_1)^{-1} - S_0^{-1}]P_1 + [(S_0 + S_1 + S_2)^{-1} - (S_0 + S_1)^{-1}]P_2 + [(S_0 + S_1 + S_2 + S_3)^{-1} - (S_0 + S_1 + S_2)^{-1}]P_3 + [(S_0 + S_1 + S_2 + S_3 + S_4)^{-1} - (S_0 + S_1 + S_2 + S_3)^{-1}]P_4,$$

Let us compute the following:

$$\begin{aligned} SK &= S_0S_0^{-1} + [S_0(S_0 + S_1)^{-1} - S_0S_0^{-1} + S_1S_0^{-1} + S_1(S_0 + S_1)^{-1} - S_1S_0^{-1}]P_1 \\ &+ [S_0(S_0 + S_1 + S_2)^{-1} - S_0(S_0 + S_1)^{-1} + S_1(S_0 + S_1 + S_2)^{-1} - S_1(S_0 + S_1)^{-1} \\ &+ S_2(S_0 + S_1 + S_2)^{-1} - S_2(S_0 + S_1)^{-1} + S_2S_0^{-1} + S_2(S_0 + S_1)^{-1} - S_2S_0^{-1}]P_2 + \\ &[S_0(S_0 + S_1 + S_2 + S_3)^{-1} - S_0(S_0 + S_1 + S_2)^{-1} + S_1(S_0 + S_1 + S_2 + S_3)^{-1} - S_1(S_0 + S_1 + S_2)^{-1} \\ &+ S_2(S_0 + S_1 + S_2 + S_3)^{-1} - S_2(S_0 + S_1 + S_2)^{-1} + S_3(S_0 + S_1 + S_2 + S_3)^{-1} \\ &- S_3(S_0 + S_1 + S_2)^{-1} + S_3S_0^{-1} + S_3(S_0 + S_1)^{-1} - S_3S_0^{-1} + S_3(S_0 + S_1 + S_2 + S_3)^{-1} \\ &- S_3(S_0 + S_1)^{-1}]P_3 + \\ &[S_0(S_0 + S_1 + S_2 + S_3 + S_4)^{-1} - S_0(S_0 + S_1 + S_2 + S_3)^{-1} + S_1(S_0 + S_1 + S_2 + S_3 + S_4)^{-1} \\ &- S_1(S_0 + S_1 + S_2 + S_3)^{-1} + S_2(S_0 + S_1 + S_2 + S_3 + S_4)^{-1} - S_2(S_0 + S_1 + S_2 + S_3)^{-1} \\ &+ S_3(S_0 + S_1 + S_2 + S_3 + S_4)^{-1} - S_3(S_0 + S_1 + S_2 + S_3)^{-1} + S_4S_0^{-1} + S_4(S_0 + S_1)^{-1} \\ &- S_4S_0^{-1} + S_4(S_0 + S_1 + S_2)^{-1} - S_4(S_0 + S_1)^{-1} + S_4(S_0 + S_1 + S_2 + S_3)^{-1} \\ &- S_4(S_0 + S_1 + S_2)^{-1} + S_4(S_0 + S_1 + S_2 + S_3 + S_4)^{-1} - S_4(S_0 + S_1 + S_2 + S_3)^{-1}]P_4 = \\ &U_{n \times n} + [O_{n \times n}]P_1 + [O_{n \times n}]P_2 + [O_{n \times n}]P_3 + [O_{n \times n}]P_4 = U_{n \times n} \end{aligned}$$

Thus, S is invertible and $S^{-1} = K$.

For the converse, if we suppose that S is invertible, then there exists $K = K_0 + K_1P_1 + K_2P_2 + K_3P_3 + K_4P_4 \in 4 - SP_M$ such that $S \times K = U_{n \times n}$. This is equivalent to the following:

$$\begin{cases} S_0K_0 = U_{n \times n} & (9) \\ S_0K_1 + S_1K_0 + S_1K_1 = O_{n \times n} & (10) \\ S_0K_2 + S_1K_2 + S_2K_2 + S_2K_1 + S_2K_0 = O_{n \times n} & (11) \\ S_0K_3 + S_1K_3 + S_2K_3 + S_3K_3 + S_3K_0 + S_3K_1 + S_3K_2 = O_{n \times n} & (12) \\ S_0K_4 + S_1K_4 + S_2K_4 + S_3K_4 + S_4K_0 + S_4K_1 + S_4K_2 + S_4K_3 + S_4K_4 = O_{n \times n} & (13) \end{cases}$$

Equation (9) implies that S_0 is invertible and $S_0^{-1} = K_0$.

By adding (9) to (10), we obtain $(S_0 + S_1)(K_0 + K_1) = U_{n \times n}$, so that $S_0 + S_1$ is invertible and $(S_0 + S_1)^{-1} = K_0 + K_1$, hence $K_1 = (S_0 + S_1)^{-1} - S_0^{-1}$.

By adding (9) to (10) to (11), we obtain $(S_0 + S_1 + S_2)(K_0 + K_1 + K_2) = U_{n \times n}$, so that $S_0 + S_1 + S_2$ is invertible and $(S_0 + S_1 + S_2)^{-1} = K_0 + K_1 + K_2$, hence $K_2 = (S_0 + S_1 + S_2)^{-1} - (S_0 + S_1)^{-1}$.

By adding (9) to (10) to (11) to (12), we obtain $(S_0 + S_1 + S_2 + S_3)(K_0 + K_1 + K_2 + K_3) = U_{n \times n}$, so that $S_0 + S_1 + S_2 + S_3$ is invertible and $(S_0 + S_1 + S_2 + S_3)^{-1} = K_0 + K_1 + K_2 + K_3$, hence $K_3 = (S_0 + S_1 + S_2 + S_3)^{-1} - (S_0 + S_1 + S_2)^{-1}$.

By adding all equations, we obtain the following:

$$(S_0 + S_1 + S_2 + S_3 + S_4)(K_0 + K_1 + K_2 + K_3 + K_4) = U_{n \times n}$$

so that $S_0 + S_1 + S_2 + S_3 + S_4$ is invertible and $(S_0 + S_1 + S_2 + S_3 + S_4)^{-1} = K_0 + K_1 + K_2 + K_3 + K_4$; hence $K_4 = (S_0 + S_1 + S_2 + S_3 + S_4)^{-1} - (S_0 + S_1 + S_2 + S_3)^{-1}$.

(3). For $n = 1$, it holds directly. We suppose that it is true. For $n = k$, we must prove it for $k + 1$.

$$\begin{aligned}
 S^{k+1} = SS^k = S_0S_0^k &+ \left[S_0(S_0 + S_1)^k - S_0S_0^k + S_1S_0^k + S_1(S_0 + S_1)^k - S_1S_0^k \right] P_1 \\
 &+ \left[S_0(S_0 + S_1 + S_2)^k - S_0(S_0 + S_1)^k + S_1(S_0 + S_1 + S_2)^k - S_1(S_0 + S_1)^k + S_2(S_0 + S_1 + S_2)^k \right. \\
 &\quad \left. - S_2(S_0 + S_1)^k + S_2S_0^k + S_2(S_0 + S_1)^k - S_2S_0^k \right] P_2 \\
 &+ \left[S_0(S_0 + S_1 + S_2 + S_3)^k - S_0(S_0 + S_1 + S_2)^k + S_1(S_0 + S_1 + S_2 + S_3)^k - S_1(S_0 + S_1 + S_2)^k \right. \\
 &\quad \left. + S_2(S_0 + S_1 + S_2 + S_3)^k - S_2(S_0 + S_1 + S_2)^k + S_3(S_0 + S_1 + S_2 + S_3)^k - S_3(S_0 + S_1 + S_2)^k \right. \\
 &\quad \left. + S_3S_0^k + S_3(S_0 + S_1)^k - S_3S_0^k + S_3(S_0 + S_1 + S_2 + S_3)^k - S_3(S_0 + S_1)^k \right] P_3 + \\
 &\left[S_0(S_0 + S_1 + S_2 + S_3 + S_4)^k - S_0(S_0 + S_1 + S_2 + S_3)^k + S_1(S_0 + S_1 + S_2 + S_3 + S_4)^k - \right. \\
 &\quad S_1(S_0 + S_1 + S_2 + S_3)^k + S_2(S_0 + S_1 + S_2 + S_3 + S_4)^k - S_2(S_0 + S_1 + S_2 + S_3)^k + S_3(S_0 + S_1 + \\
 &\quad S_2 + S_3 + S_4)^k - S_3(S_0 + S_1 + S_2 + S_3)^k + S_4S_0^k + S_4(S_0 + S_1)^k - S_4S_0^k + S_4(S_0 + S_1 + S_2)^k - \\
 &\quad S_4(S_0 + S_1)^k + S_4(S_0 + S_1 + S_2 + S_3)^k - S_4(S_0 + S_1 + S_2)^k + S_4(S_0 + S_1 + S_2 + S_3 + S_4)^k - \\
 &\quad \left. S_4(S_0 + S_1 + S_2 + S_3)^k \right] P_4 = S_0^{k+1} + \left[(S_0 + S_1)^{k+1} - S_0S_0^{k+1} \right] P_1 + \left[(S_0 + S_1 + S_2)^{k+1} - (S_0 + \right. \\
 &\quad \left. S_1)^{k+1} \right] P_2 + \left[(S_0 + S_1 + S_2 + S_3)^{k+1} - (S_0 + S_1 + S_2)^{k+1} \right] P_3 + \left[(S_0 + S_1 + S_2 + S_3 + S_4)^n - \right. \\
 &\quad \left. (S_0 + S_1 + S_2 + S_3)^n \right] P_4.
 \end{aligned}$$

This is so that the proof holds by induction.

$$\begin{aligned}
 (4). e^S = \sum_{n=0}^{\infty} \frac{S^n}{n!} &= \sum_{n=0}^{\infty} \frac{S_0^n}{n!} + \left[\sum_{n=0}^{\infty} \frac{(S_0+S_1)^n}{n!} - \sum_{n=0}^{\infty} \frac{S_0^n}{n!} \right] P_1 + \left[\sum_{n=0}^{\infty} \frac{(S_0+S_1+S_2)^n}{n!} \right. \\
 &\quad \left. - \sum_{n=0}^{\infty} \frac{(S_0+S_1)^n}{n!} \right] P_2 + \left[\sum_{n=0}^{\infty} \frac{(S_0+S_1+S_2+S_3)^n}{n!} - \sum_{n=0}^{\infty} \frac{(S_0+S_1+S_2)^n}{n!} \right] P_3 + \left[\sum_{n=0}^{\infty} \frac{(S_0+S_1+S_2+S_3+S_4)^n}{n!} \right. \\
 &\quad \left. - \sum_{n=0}^{\infty} \frac{(S_0+S_1+S_2+S_3)^n}{n!} \right] P_4 = e^{S_0} + [e^{S_0+S_1} - e^{S_0}] P_1 + [e^{S_0+S_1+S_2} - e^{S_0+S_1}] P_2 + [e^{S_0+S_1+S_2+S_3} \\
 &\quad - e^{S_0+S_1+S_2}] P_3 + [e^{S_0+S_1+S_2+S_3+S_4} - e^{S_0+S_1+S_2+S_3}] P_4. \quad \square
 \end{aligned}$$

Definition 7. Let $S = S_0 + S_1P_1 + S_2P_2 + S_3P_3 + S_4P_4 \in 4 - SP_M$, then we define the following:

$$\det(S) = \det(S_0) + [\det(S_0 + S_1) - \det(S_0)]P_1 + [\det(S_0 + S_1 + S_2) - \det(S_0 + S_1)]P_2 + [\det(S_0 + S_1 + S_2 + S_3) - \det(S_0 + S_1 + S_2)]P_3 + [\det(S_0 + S_1 + S_2 + S_3 + S_4) - \det(S_0 + S_1 + S_2 + S_3)]P_4.$$

Theorem 6. S is invertible if, and only if, $\det(S)$ is invertible in $4 - SP_R$.

The proof holds via a similar discussion of the 3-plithogenic case.

3.5. Symbolic 4-Plithogenic Eigen Values/Vectors

Theorem 7. Let $S = S_0 + S_1P_1 + S_2P_2 + S_3P_3 + S_4P_4 \in 4 - SP_M$, then $A = a_0 + a_1P_1 + a_2P_2 + a_3P_3 + a_4P_4$ is an eigen value of S if, and only if, a_0 eigen value of S_0 , $a_0 + a_1$ eigen value of $S_0 + S_1$, $a_0 + a_1 + a_2$ eigen value of $S_0 + S_1 + S_2$, $a_0 + a_1 + a_2 + a_3$ eigen value of $S_0 + S_1 + S_2 + S_3$, $a_0 + a_1 + a_2 + a_3 + a_4$ eigen value of $S_0 + S_1 + S_2 + S_3 + S_4$.

In addition, $X = X_0 + X_1P_1 + X_2P_2 + X_3P_3 + X_4P_4$ is the corresponding eigen vector of A if, and only if, the X_0 eigen vector of a_0 , the $X_0 + X_1$ eigen vector of $a_0 + a_1$, the $X_0 + X_1 + X_2$ eigen vector of $a_0 + a_1 + a_2$, the $X_0 + X_1 + X_2 + X_3$ eigen vector of $a_0 + a_1 + a_2 + a_3$, the $X_0 + X_1 + X_2 + X_3 + X_4$ eigen vector of $a_0 + a_1 + a_2 + a_3 + a_4$.

Proof of Theorem 7. From the equation $AS = AX$, we obtain the following:

$$\left\{ \begin{aligned} & a_0S_0 = a_0X_0 \quad (14) \\ & a_0S_1 + a_1S_0 + a_1S_1 = a_0X_1 + a_1X_0 + a_1X_1 \quad (15) \\ & a_0S_2 + a_1S_2 + a_2S_2 + a_2S_1 + a_2S_0 = a_0X_2 + a_1X_2 + a_2X_2 + a_2X_1 + a_2X_0 \quad (16) \\ & a_0S_3 + a_1S_3 + a_2S_3 + a_3S_3 + a_3S_0 + a_3S_1 + a_3S_2 = a_0X_3 + a_1X_3 + a_2X_3 + a_3X_3 + a_3X_0 + a_3X_1 + a_3X_2 \quad (17) \\ & a_0S_4 + a_1S_4 + a_2S_4 + a_3S_4 + a_4S_0 + a_4S_1 + a_4S_2 + a_4S_3 + a_4S_4 = a_0X_4 + a_1X_4 + a_2X_4 + a_3X_4 + a_4X_0 + a_4X_1 + a_4X_2 + a_4X_3 + a_4X_4 \quad (18) \end{aligned} \right.$$

Equation (14) implies that a_0 is the eigen value of S_0 , with X_0 as the eigen vector.

By adding (14) to (15), we obtain $(a_0 + a_1)(S_0 + S_1) = (a_0 + a_1)(X_0 + X_1)$, which means that $a_0 + a_1$ is the eigen value of $S_0 + S_1$, with $X_0 + X_1$ as the eigen vector.

By adding (14) to (15) to (16), we obtain $(a_0 + a_1 + a_2)(S_0 + S_1 + S_2) = (a_0 + a_1 + a_2)(X_0 + X_1 + X_2)$, which means that $a_0 + a_1 + a_2$ is the eigen value of $S_0 + S_1 + S_2$, with $X_0 + X_1 + X_2$ as the eigen vector.

By adding (14) to (15) to (16) to (17), we obtain $(a_0 + a_1 + a_2 + a_3)(S_0 + S_1 + S_2 + S_3) = (a_0 + a_1 + a_2 + a_3)(X_0 + X_1 + X_2 + X_3)$, which means that $a_0 + a_1 + a_2 + a_3$ is the eigen value of $S_0 + S_1 + S_2 + S_3$, with $X_0 + X_1 + X_2 + X_3$ as the eigen vector.

By adding all equations, we obtain the following: $(a_0 + a_1 + a_2 + a_3 + a_4)(S_0 + S_1 + S_2 + S_3 + S_4) = (a_0 + a_1 + a_2 + a_3 + a_4)(X_0 + X_1 + X_2 + X_3 + X_4)$, which means that $a_0 + a_1 + a_2 + a_3 + a_4$ is the eigen value of $S_0 + S_1 + S_2 + S_3 + S_4$, with $X_0 + X_1 + X_2 + X_3 + X_4$ as the eigen vector.

This is so that the proof is complete. \square

Example 6. Consider the following matrix:

$$S = \begin{pmatrix} 3 + 2P_1 - P_3 - P_4 & 4 + P_2 - 3P_3 \\ 2P_1 + P_2 + P_3 - P_4 & 5 + P_1 + P_4 \end{pmatrix} = \begin{pmatrix} 3 & 4 \\ 0 & 5 \end{pmatrix} + \begin{pmatrix} 2 & 0 \\ 2 & 1 \end{pmatrix}P_1 + \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}P_2 + \begin{pmatrix} -1 & -3 \\ 1 & 0 \end{pmatrix}P_3 + \begin{pmatrix} -1 & 0 \\ -1 & 1 \end{pmatrix}P_4 = A_0 + A_1P_1 + A_2P_2 + A_3P_3 + A_4P_4$$

The eigen values of $A_0 = \begin{pmatrix} 3 & 4 \\ 0 & 5 \end{pmatrix}$ are $\{3, 5\}$.

The eigen values of $A_0 + A_1 = \begin{pmatrix} 2 & 0 \\ 2 & 1 \end{pmatrix}$ are $\left\{ \frac{11 + \sqrt{33}}{2}, \frac{11 - \sqrt{33}}{2} \right\}$.

The eigen values of $A_0 + A_1 + A_2 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ are $\left\{ \frac{11 + \sqrt{61}}{2}, \frac{11 - \sqrt{61}}{2} \right\}$.

The eigen values of $A_0 + A_1 + A_2 + A_3 = \begin{pmatrix} -1 & -3 \\ 1 & 0 \end{pmatrix}$ are $\{8, 2\}$.

The eigen values of $A_0 + A_1 + A_2 + A_3 + A_4 = \begin{pmatrix} -2 & -3 \\ 0 & 1 \end{pmatrix}$ are $\{-2, 1\}$.

The eigen values of the symbolic 4-plithogenic matrix A are as follows:

$$U_1 = 3 + \left(\frac{11 + \sqrt{33}}{2} - 3 \right)P_1 + \left(\frac{11 + \sqrt{61}}{2} - \frac{11 + \sqrt{33}}{2} \right)P_2 + \left(8 - \frac{11 + \sqrt{61}}{2} \right)P_3 + (-10)P_4$$

$$U_2 = 3 + \left(\frac{11 + \sqrt{33}}{2} - 3 \right)P_1 + \left(\frac{11 + \sqrt{61}}{2} - \frac{11 + \sqrt{33}}{2} \right)P_2 + \left(2 - \frac{11 + \sqrt{61}}{2} \right)P_3 + (-4)P_4$$

$$U_3 = 3 + \left(\frac{11 + \sqrt{33}}{2} - 3 \right)P_1 + \left(\frac{11 - \sqrt{61}}{2} - \frac{11 + \sqrt{33}}{2} \right)P_2 + \left(8 - \frac{11 + \sqrt{61}}{2} \right)P_3 + (-7)P_4$$

$$U_4 = 3 + \left(\frac{11 + \sqrt{33}}{2} - 3 \right)P_1 + \left(\frac{11 - \sqrt{61}}{2} - \frac{11 + \sqrt{33}}{2} \right)P_2 + \left(2 - \frac{11 + \sqrt{61}}{2} \right)P_3 + (-1)P_4$$

$$U_5 = 3 + \left(\frac{11 - \sqrt{33}}{2} - 3 \right)P_1 + \left(\frac{11 + \sqrt{61}}{2} - \frac{11 - \sqrt{33}}{2} \right)P_2 + \left(8 - \frac{11 + \sqrt{61}}{2} \right)P_3 + (-10)P_4$$

$$\begin{aligned}
 U_6 &= 3 + \left(\frac{11 - \sqrt{33}}{2} - 3\right)P_1 + \left(\frac{11 - \sqrt{61}}{2} - \frac{11 - \sqrt{33}}{2}\right)P_2 + \left(2 - \frac{11 - \sqrt{61}}{2}\right)P_3 + (-4)P_4 \\
 U_7 &= 3 + \left(\frac{11 - \sqrt{33}}{2} - 3\right)P_1 + \left(\frac{11 - \sqrt{61}}{2} - \frac{11 - \sqrt{33}}{2}\right)P_2 + \left(8 - \frac{11 - \sqrt{61}}{2}\right)P_3 + (-7)P_4 \\
 U_8 &= 3 + \left(\frac{11 - \sqrt{33}}{2} - 3\right)P_1 + \left(\frac{11 + \sqrt{61}}{2} - \frac{11 - \sqrt{33}}{2}\right)P_2 + \left(2 - \frac{11 - \sqrt{61}}{2}\right)P_3 + (-1)P_4 \\
 U_9 &= 5 + \left(\frac{11 + \sqrt{33}}{2} - 5\right)P_1 + \left(\frac{11 + \sqrt{61}}{2} - \frac{11 + \sqrt{33}}{2}\right)P_2 + \left(8 - \frac{11 + \sqrt{61}}{2}\right)P_3 + (-10)P_4 \\
 U_{10} &= 5 + \left(\frac{11 + \sqrt{33}}{2} - 5\right)P_1 + \left(\frac{11 + \sqrt{61}}{2} - \frac{11 + \sqrt{33}}{2}\right)P_2 + \left(2 - \frac{11 + \sqrt{61}}{2}\right)P_3 + (-4)P_4 \\
 U_{11} &= 5 + \left(\frac{11 + \sqrt{33}}{2} - 5\right)P_1 + \left(\frac{11 - \sqrt{61}}{2} - \frac{11 + \sqrt{33}}{2}\right)P_2 + \left(8 - \frac{11 - \sqrt{61}}{2}\right)P_3 + (-7)P_4 \\
 U_{12} &= 5 + \left(\frac{11 + \sqrt{33}}{2} - 5\right)P_1 + \left(\frac{11 - \sqrt{61}}{2} - \frac{11 + \sqrt{33}}{2}\right)P_2 + \left(2 - \frac{11 - \sqrt{61}}{2}\right)P_3 + (-1)P_4 \\
 U_{13} &= 5 + \left(\frac{11 - \sqrt{33}}{2} - 5\right)P_1 + \left(\frac{11 + \sqrt{61}}{2} - \frac{11 - \sqrt{33}}{2}\right)P_2 + \left(8 - \frac{11 + \sqrt{61}}{2}\right)P_3 + (-10)P_4 \\
 U_{14} &= 5 + \left(\frac{11 - \sqrt{33}}{2} - 5\right)P_1 + \left(\frac{11 + \sqrt{61}}{2} - \frac{11 - \sqrt{33}}{2}\right)P_2 + \left(2 - \frac{11 + \sqrt{61}}{2}\right)P_3 + (-4)P_4 \\
 U_{15} &= 5 + \left(\frac{11 - \sqrt{33}}{2} - 5\right)P_1 + \left(\frac{11 - \sqrt{61}}{2} - \frac{11 - \sqrt{33}}{2}\right)P_2 + \left(8 - \frac{11 - \sqrt{61}}{2}\right)P_3 + (-7)P_4 \\
 U_{16} &= 5 + \left(\frac{11 - \sqrt{33}}{2} - 5\right)P_1 + \left(\frac{11 + \sqrt{61}}{2} - \frac{11 - \sqrt{33}}{2}\right)P_2 + \left(2 - \frac{11 - \sqrt{61}}{2}\right)P_3 + (-1)P_4.
 \end{aligned}$$

We continue using the same process to obtain all other eigen values.

Remark 5. If A has n eigen values for $0 \leq i \leq s$, then A has n^5 eigen values.

3.6. Symbolic 4-Plithogenic Diagonalization

Theorem 8. Let $T = T_0 + T_1P_1 + T_2P_2 + T_3P_3 + T_4P_4$ be an n -square symbolic 4-plithogenic matrix, then T is diagonalizable if, and only if, $T_0, T_0 + T_1, T_0 + T_1 + T_2, T_0 + T_1 + T_2 + T_3, T_0 + T_1 + T_2 + T_3 + T_4$ are diagonalizable.

Proof of Theorem 8. T are diagonalizable if, and only if, there exists $L = L_0 + L_1P_1 + L_2P_2 + L_3P_3 + L_4P_4$ and $D = D_0 + D_1P_1 + D_2P_2 + D_3P_3 + D_4P_4$, such that $T = L^{-1}DL$, where D is diagonal and L is invertible.

The equation $T = L^{-1}DL$ is equivalent to the following:

$$\left\{ \begin{aligned}
 &T_0 = L_0^{-1}D_0L_0 \\
 &T_0 + T_1 = (L_0 + L_1)^{-1}(D_0 + D_1)(L_0 + L_1) \\
 &T_0 + T_1 + T_2 = (L_0 + L_1 + L_2)^{-1}(D_0 + D_1 + D_2)(L_0 + L_1 + L_2) \\
 &T_0 + T_1 + T_2 + T_3 = (L_0 + L_1 + L_2 + L_3)^{-1}(D_0 + D_1 + D_2 + D_3)(L_0 + L_1 + L_2 + L_3) \\
 &T_0 + T_1 + T_2 + T_3 + T_4 = (L_0 + L_1 + L_2 + L_3 + L_4)^{-1}(D_0 + D_1 + D_2 + D_3 + D_4)(L_0 + L_1 + L_2 + L_3 + L_4)
 \end{aligned} \right.$$

This implies the proof. \square

4. Conclusions

In this scientific work, we have studied the symbolic 3-plithogenic square real matrices and the symbolic 4-plithogenic square real matrices from many different perspectives, and we have presented many algorithms and theorems to describe the invertibility, the eigen values, vectors, and diagonalization of these matrices by transforming them into classical matrices. In the future, we aim to study the symbolic 5-plithogenic and 6-plithogenic real square matrices in a similar approach to that used for the symbolic 3-plithogenic matrices and symbolic 4-plithogenic matrices.

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