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On Smarandache Semigroups

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Abstract - In this article we present some basis definitions about semigrous, groups and Smarandache semigroups. And the end we give the characterization of Smarandache semigroups and Smarandache semigroups where any Smarandache subset is a subsemigroup.of Smarandache semigroups and Smarandache semigroups where any Smarandache subset is a subsemigroup.

Keywords—Semigroup, group, cyclic semigroup, cyclic group, semigroup of idempotents, periodic semigroup, nilpotent semigroup, Smarandache semigroup.

I. INTRODUCTION

Smarandache notions, which can be undoubtedly characterized as interesting mathematics, has the capacity of being utilized to analyse, study and introduce, naturally, the concepts of several structures by means of extension or identification as a substructure.

Generally, in any human field, a Smarandache Structure on a set A means a weak structure W on A such that there exists a proper subset $B \subset A$ which is embedded with a stronger structure S.

The Smarandache semigroups exhibit properties of both a group and a semigroup simultaneously. In this section, we just recall the definitions of semigroups, groups and Smarandache semigroups.

Semigroups are the algebraic structures in which are defined a binary operation which is both closed and associative. A notion of the Smarandache semigroup was introduce in [1].

II. BASIC DEFINITIONS

Definition 1. Let S be a set on which is defined a binary operation $\circ$, a couple $< S, \circ >$ is a semigroup if for all a, b $\in$ S we have $a \circ b = c \in S$ (closed) and $a \circ (b \circ c) = (a \circ b) \circ c$ for all a, b, c $\in$ S (associative).

Definition 2. Let G be a non empty set on which is defined a binary operation $\circ$, a couple $< G, \circ >$ is a group if for all a, b, c $\in$ G operation $\circ$ is closed in G ($a \circ b \in G$); is associative ($a \circ (b \circ c) = (a \circ b) \circ c$); there exists an identity element $e \in G$ that for all $a \in G$ $a \circ e = e \circ a = a$; for every element $a \in G$ there exists an inverse element $a^{-1} \in G$ such that $a \circ a^{-1} = a^{-1} \circ a = e$.

Definition 3. Let $< S, \circ >$ be a semigroup. A subset H of S is said to be a subsemigroup of S if H itself is a semigroup under the operations of S.

Definition 4. A non empty subset G of a semigroup S is said to be a subgroup of S if, under the product in S, G itself forms a group.

For every group G with identity element e the set {e} is a subgroup of G which we call as trivial or improper subgroup of G. Likewise G the group itself is a subgroup of G called the improper subgroup of G. So H a subset of G is called a proper subgroup of G if H is not the identity subgroup or H is not the whole group G.

Definition 5. Let S be a semigroup. An element $x \in S$ is said to be an idempotent if $x \circ x = x$. If all elements of S are idempotent S is said to be a semigroup of idempotents or band.

Definition 6. Let S be a semigroup. An element $0 \in S$ is said to be zero if for every $x \in S$ $0 \circ x = x \circ 0 = 0$. An element x of a semigroup S with zero 0 is said to be nilelement if exists n that $x^n = 0$. A semigroup S is a nilsemigroup if every element of S is a nilelement. A semigroup S is a nilpotent semigroup if exists n that $S^n = 0$. The smallest number n with this property is called a degree of nilpotent semigroup S.

Definition 7. Let S be a semigroup**.** Cardinal |S| of a semigroup S is named order of S.

Definition 8. A semigroup S is said to be combinatorial semigroup if every subgroup of S is a singleton.

Obviously that every semigroup of idempotents, every nilpotent semigroup and every nilsemigroup are combinatorial semigroups.

Definition 9. Semigroup S is named cyclic semigroup if it is generated by an element $a \in S$

(i.e, $S = \{a, a^2, \ldots, a^n, \ldots\}$) and this fact is denoted by

$S = < a >$. There exist two cases:

1) S is an infinite;
2) S is a finite $S = \{a, a^2, \ldots, a^k, \ldots, a^{k+n-1}\}$ where $a^k = a^{k+n}$. In this case S is named the cyclic semigroup of type (k, n).

The number k is named an index of S and the number n is named a period of S. In this case S has unique idempotent e; the set Se = $\{a^k, \ldots, a^{k+n-1}\}$ is a greatest cyclic subgroup of order n and ideal of S.

Definition 10. Group G is named cyclic group if it is generated by an element a $\in$ G (i.e, G = $\{a^n \mid n = 0, \pm 1, \pm 2, \ldots\}$) and this fact is denoted by G = < a >. There exist two cases:

1) G is an infinite. In this case G is isomorphic the group Z of integer numbers under operation of addition;
2) G is a finite. In this case G is isomorphic the group $Z_n = \{0, 1, \ldots, n-1\}$ under operation of addition by mod n.

Definition 11. The Smarandache semigroup (S-semigroup) is defined to be a semigroup S such that S has a proper subgroup G (i.e., G is a group with respect to the induced operation, G $\subset$ S and G have at least two elements).

Example 1. Let Z be the group of integer numbers. The group 2Z of even integer numbers is the proper subgroup of Z.

It means that infinite cyclic group is the Smarandache semigroup (S-semigroup). On the other hand it is clear that infinite cyclic semigroup is not a Smarandache semigroup (is not a S-semigroup).

Definition 12. Let S be a S-semigroup. A proper subset A of S is said to be a Smarandache subsemigroup of S if A itself is a S-semigroup, that is A is a semigroup of S containing a proper subset B such that B is the group under the operations of S.

Definition 13. Let S be a S-semigroup. If A $\subset$ S is a proper subset of S and A is a subgroup which cannot be contained in any proper subsemigroup of S we say A is the largest subgroup of S.

Definition 14. Let S be a S-semigroup. If A be a proper subset of S which is a subsemigroup of S and A contains the largest group of S then we say A to be the Smarandache hyper subsemigroup of S.

Definition 15. Let S be a S-semigroup. We say S is a Smarandache simple semigroup if S has no proper subsemigroup A, which contains the largest subgroup of S or S has no Smarandache hyper subsemigroup.

The next definition is new.

Definition 16. Let S be a S-semigroup. We say S is a Smarandache strongly simple semigroup if S has no proper Smarandache subsemigroup.

Let S be a semigroup and 0 $\notin$ S. Let consider the semigroup

$S^0 = \begin{cases} S, & \text{if } S \text{ has zero and } |S| \geq 2, \\ S \cup \{0\} & \text{otherwise} \end{cases}$ where $0 \circ 0 = 0 \circ a = a \circ 0 = 0$ for any a $\in$ S.

Definition 17. The semigroup S^0 is named a semigroup adjoint with zero.

The next definition is new.

Definition 18. Let S be a S-semigroup. We say that a proper subset A $\subset$ S is a Smarandache subset if A has a proper subgroup.

Definition 19. Let S be a semigroup. We say that S is a periodic semigroup if any cyclic subsemigroup of S is a finite.

Definition 20. Let S be a periodic semigroup and E_S be a set of idempotents of S. For any e $\in$ E_S a set
$K_e = \{x \in S \mid x^n = e \text{ for some } n\}$
is named a class of torsion of the semigroup S corresponding an idempotent e.

The set $G_e = K_e e$ is an ideal of < K_e > and the greatest subgroup of K_e. A periodic semigroup S is a disjunctive union of their classes of torsion ($S = \bigcup K_e$ ($e \in E_S$) $K_i \cap K_j = \varnothing$ iff $i \neq j$)

Supplementary reeding about groups, semigroups and Smarandache semigroups can be found in [2], [3], [4], [5] and [6].

III. MAIN RESULTS

Now we will give the description of finite Smarandache semigroups.

Theorem 1. Let S be a finite semigroup. The semigroup S is a Smarandache semigroup if and only if S is not a combinatorial semigroup or S is not a cyclic group of prime order.

Proof. Let S be a cyclic group of prime order. By Theorem of Lagrange S has no proper subgroups therefore S is not Smarandache semigroup.

Let S be a combinatorial semigroup then by definition S is not Smarandache semigroup.

Let S not be a cyclic group of prime order and not be a combinatorial semigroup then it is clear that S has a proper subgroup and be a Smarandache semigroup.

The description of infinite Smarandache semigroup gives Theorem 2.

Theorem 2. Let S be an infinite semigroup. Semigroup S is a Smarandache semigroup if and only if S has an idempotent and not be a combinatorial semigroup.

Proof. Let S be a semigroup without idempotents then S has no subgroups in particular has no proper subgroups. If S is not a combinatorial semigroup then we have two cases.

The first, S is the infinite cyclic group then S is a Smarandache semigroup
(see Example 1).

The second, S is not a cyclic group. But if S is a group then S has proper subgroups and S is a Smarandache semigroup. If S is not a group and is not combinatorial semigroup then S have a proper subgroup and S is a Smarandache semigroup.

The next theorem is a generalization of theorem 4.5.3 [2].

Theorem 3. Let G be a group and $|G| \geq 2$. Then a semigroup $S = G^0$ is a Smarandache simple semigroup.

Proof. The semigroup $S = G^0$ is a Smarandache semigroup, because $G \subset S$ and G is a proper subgroup of S. The group G is the largest subgroup of S. Any proper subsemigroup of S not contains G. Then S has no Smarandache hyper subsemigroup. Then S is a Smarandache simple semigroup.

Theorem 4. Let G be a cyclic group of prime order, then $S = G^0$ is a Smarandache strongly simple semigroup.

Proof. Let G be a cyclic group of prime order, and $S = G^0$. In this case any proper subsemigroup of S is equal to $S' = A \cup \{0\}$ where $A \subset G$ or $S' \subseteq G$. If $S' = A \cup \{0\}$ where $A \subset G$ then S' don't has proper subgroup because G has only two subgroups G and {e} where e is an identity element of G and S' is not a Smarandache semigroup. If $S' \subseteq G$ then S' also don't has a proper subgroup and is not a Smarandache semigroup. Then S is a Smarandache strongly simple semigroup.

Now we will study the Smarandache semigroups where any Smarandache subset is a Smarandache subsemigroup. These semigroups we will denote by S^3-semigroups.

Theorem 5. Let S be a S^3-semigroup then S is a periodic semigroup.

Proof. Let S be a S^3-semigroup and G is a proper subgroup of S. In this case exists cyclic subgroup G_1 of S.

If G_1 is an infinite i.e.
$G_1 = \{a^n| n = 0, \pm1, \pm2, \dots\}$ then exists subgroup
$G_2 = \{a^n| n = 0, \pm2, \pm4, \dots\}$ ($G_2 \subset G_1$). The set
$X = \{a\} \cup G_2$ is a Smarandache subset of S but is not subsemigroup.

Let G_1 be a finite group; $x \in S$ and cyclic subsemigroup $< x >$ is an infinite then the set $X = \{x\} \cup G_1$ is a Smarandache subset of S but is not subsemigroup.

Theorem 6. Let S be a periodic semigroup and $G_e \neq \{e\}$ is a greatest subgroup of K_e ($e \in E_S$). If S is a S^3-semigroup then Ge is a cyclic group of a prime order.

Proof. Assume that is not a cyclic group of prime order. Then exists subgroup $G_1 \subset G_e$ where $|G_1| \geq 2$. It means that G_e is a S-semigroup. Let $x \in G_e \setminus G_1$ then $\{x\} \cup G_1$ is a S-subset and consequently is a subsemigroup and in particularly is a subgroup. Then $x^{-1} \in \{x\} \cup G_1$. If $x^{-1} \in G_1$ then $x \in G_1$ it is a contradiction. It means that $x^{-1} = x$. It is clear that $x \neq x^2$ then $x^2 \in G_1$, $x^2x^{-1} = xe = x = x^{-1}$, $x^2x = xx^{-1}x = xe = x$. Consequently, $< x > = \{x, e\}$ is a subgroup. Assume that $y \in G_1$, $y \neq x$, $y \neq e$ then {x, y, e} is a S-subset and is a subsemigroup and in particularly is a subgroup. As $y \in G_1 \setminus < x >$ then $y^{-1} \neq x$, $y^{-1} \neq e$, then $y^{-1} = y$. It means that xy = x or xy = y or xy = e.

If xy = x and yx = y then yxy = yx = y and yxy = yy = y^2 = e and y = e it is a contradiction.

If xy = yx = x then xyx = x^2 = e and yxyx = ye = y, on the other hand yxyx = x^2 = e and y = e it is a contradiction.

If xy = yx = e then xyy = x and xyy = y and x = y it is a contradiction.

If xy = x and yx = e then yyx = ex = x and yyx = ye =y; x = y it is a contradiction.

Other cases also lead to a contradictionю

Therefore, does not exist $x \in G_e \setminus G_1$ and G_e is a cyclic group of prime order.

Theorem 7. Let S be a periodic S^3-semigroup, $x \in K_e \setminus G_e$ where $e \in E_S$ and $|G_e| \geq 2$. Then
$x^2 \in G_e$.

Proof. The set $\{x\} \cup G_e$ is a S-subconjunto and therefore is a subsemigroup. Then $x^2 \in \{x\} \cup G_e$. But $x \neq x^2$ then
$x^2 \in G_e$.

Theorem 8. Let S be a periodic S^3-semigroup, $x, y \in K_e \setminus G_e$ where $e \in E_S$ and $|G_e| \geq 2$ and $x \neq y$. Then $xy, yx \in G_e$.

Proof. If xy = x then $xy^2 = (xy)y = xy = x$. On the other hand $xy^2 \in G_e$ as $y^2 \in G_e$ and $x \in G_e$. It is a contradiction. Then $xy \neq x$ and $xy \in G_e$. Similarly, $yx \in G_e$.

From theorems 7 and 8 follows

Theorem 9. Let S be a periodic S^3-semigroup, $G_e \neq \{e\}$ and $G_e \subset K_e$ where $e \in E_S$. Then K_e is a semigroup and Rees factor-semigroup K_e / G_e is a nilpotent semigroup of degree 2.

Theorem 10. Let S be a periodic S^3-semigroup $x \notin K_e$ and $|G_e| \geq 2$. Then $x = x^2$.

Proof. The set $\{x\} \cup G_e$ is a S-subset and is a subsemigroup then $x^2 \in \{x\} \cup G_e$ but $x \in K_i$ where $i \neq e$ and
$i \in E_S$. Then $x^2 \in K_i$ and $x = x^2$ is an idempotent.

Theorem 11. Let S be a periodic S^3-semigroup and i, j, e are idempotents of S pairwise different,
$G_e \neq \{e\}$. Then ij = i or ij = j or $ij \in G_e$.

Proof. The set $\{i, j, G_e\}$ is a S-subset of S then is a subsemigroup and ij = i or ij = j or $ij \in G_e$.

Theorem 12. Let S be a periodic S^3-semigroup and i, j, e are idempotents of S pairwise different,

$G_e \neq \{e\}$. Then ij = i or ij = j or ij = e.

Proof. By theorem 11 ij = i or ij = j or ij $\in G_e$.

If ij $\in G_e$ and ji = i then $(ij)^2 = i(ji)j = iij = i^2j = ij$ and ij = e.

If ij $\in G_e$ and ji = j then $(ij)^2 = i(ji)j = ijj = ij^2 = ij$ and ij = e.

If ij $\in G_e$ and ji $\in G_e$. Suppose that ij $\neq$ e then $< ij > = G_e$ and $ji = (ij)^k$ for some number k. If ij = ji then $(ij)^2 = ijji = iji = iij = ij$ and ij = e.

If ij $\neq$ ji then $(ij)^2 = i(ji)j = i(ij)^kj = (ij)^k = ji$, $(ij)^2 = ijij = ji = jij = iji = jiji = (ji)^2$. Then $(ji)^2 = ji = e$ and $(ij)^2 = ji = e$. Then ij = ije = ijji = iji = eij = jiij = jij = ijij = $(ij)^2$ = e. It is a contradiction.

Theorem 13. Let S be a semigroup. A semigroup S is a S^3-semigroup if and only if S is a periodic semigroup there exists only one idempotent e $\in E_S$ where G_e is a proper subgroup of S; the group G_e is a cyclic group of prime order; the elements of S \ K_e are idempotents and if i, j $\in E_S$ then ij = i or ij = j or
ij = e and Rees factor-semigroup K_e / G_e is a nilpotent semigroup of degree 2.

Proof. The necessity follows from theorems 5, 6, 7, 8, 9, 10, 11 and 12.
Sufficiency. Let S be a periodic semigroup and X be a S-subset of S. If X $\subseteq K_e$ then X is a subsemigroup because $X^2 \subseteq G_e$ and the Rees factor-semigroup K_e / G_e is a nilpotent semigroup of degree 2. Then S is a S^3 -semigroup.
If X $\cap$ (S \ K_e) $\neq \varnothing$ then X is a subsemigroup because $G_e \subseteq X$ and for all x, y $\in$ X, xy = x or xy = y or xy = e or xy $\in G_e$. Then S is a S^3 -semigroup.

Note that a semigroup S where any subset is a subsemigroup are exactly the semigroup of idempotents (band) where xy = x or xy = y for any x, y $\in$ S [6].

IV. CONCLUSION

In the future, we should begin to study Smarandache semirings where any Smarandace subset is a subsemiring.

REFERENCES

[1] Raul Padilha, Smarandache Algebraic Structure, *Bulletin of Pure and Applied Sciences*, Delhi, Vol. 17E, no 1, 1998, pp.119-121.

[2] Vasantha Kandasamy W.B. Smarandache Semigroups. American Research Press, Rehoboth, 2002, pp. 94.

[3] Vasantha Kandasamy W.B. Smarandache Semirings, semifields and semivector spaces. American Research Press, Rehoboth, 2002, pp. 121.

[4] Hall Marshall, Theory of Groups, New York, The Macmillian Company, 1961.

[5] Clifford A.H. and Preston G.B. The Algebraic Theory of Semigroups.Vol. 1, Mathematical Surveys. Number 7, American Mathematical Society, 1964.

[6] Shevrin L. N. and Ovsyannikov A. J. Semigroups and Their Subsemigroups Lattices; Kluver Academic Publishers, Dordrecht, 1996.