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One Modulo N Gracefullness Of Arbitrary Supersubdivisions of Graphs

V.Ramachandran

(Department of Mathematics, P.S.R Engineering College, Sevalpatti, Sivakasi, Tamil Nadu, India)

C.Sekar

(Department of Mathematics, Aditanar College of Arts and Science, Tiruchendur, Tamil Nadu, India)

E-mail: me.ram111@gmail.com, sekar.acas@gmail.com

Abstract: A function f is called a graceful labelling of a graph G with q edges if f is an injection from the vertices of G to the set $\{0, 1, 2, \ldots, q\}$ such that, when each edge xy is assigned the label $|f(x) - f(y)|$, the resulting edge labels are distinct. A graph G is said to be one modulo N graceful (where N is a positive integer) if there is a function ϕ from the vertex set of G to $\{0, 1, N, (N + 1), 2N, (2N + 1), \ldots, N(q - 1), N(q - 1) + 1\}$in such a way that (i) ϕ is $1 - 1$ (ii) ϕ induces a bijection ϕ^* from the edge set of G to $\{1, N + 1, 2N + 1, \ldots, N(q - 1) + 1\}$where $\phi^*(uv)$=$|\phi(u) - \phi(v)|$. In this paper we prove that the arbitrary supersubdivisions of paths, disconnected paths, cycles and stars are one modulo N graceful for all positive integers N.

Key Words: Modulo graceful graph, Smarandache modulo graceful graph, supersubdivisions of graphs, paths, disconnected paths, cycles and stars.

AMS(2010): 05C78

§1. Introduction

S.W.Golomb introduced graceful labelling ([1]). The odd gracefulness was introduced by R.B.Gnanajothi in [2]. C.Sekar introduced one modulo three graceful labelling ([8]) recently. V.Ramachandran and C.Sekar ([6]) introduced the concept of one modulo N graceful where N is any positive integer.In the case $N = 2$, the labelling is odd graceful and in the case $N = 1$ the labelling is graceful.We prove that the the arbitrary supersubdivisions of paths, disconnected paths, cycles and stars are one modulo N graceful for all positive integers N.

§2. Main Results

Definition 2.1 *A graph G is said to be one Smarandache modulo N graceful on subgraph $H < G$ with q edges (where N is a positive integer) if there is a function ϕ from the vertex set*

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of G to $\{0, 1, N, (N+1), 2N, (2N+1), \cdots, N(q-1), N(q-1)+1\}$ in such a way that (i) ϕ is $1-1$ (ii) ϕ induces a bijection ϕ^* from the edge set of H to $\{1, N+1, 2N+1, \cdots, N(q-1)+1\}$, and $E(G) \setminus E(h)$ to $\{1, 2, \cdots, |E(G)|-q\}$, where $\phi^*(uv)=|\phi(u)-\phi(v)|$. Particularly, if $H = G$ such a graph is said to be one modulo N graceful graph.

Definition 2.2([9]) *In the complete bipartite graph $K_{2,m}$ we call the part consisting of two vertices, the 2-vertices part of $K_{2,m}$ and the part consisting of m vertices the m-vertices part of $K_{2,m}$.Let G be a graph with p vertices and q edges. A graph H is said to be a supersubdivision of G if H is obtained by replacing every edge e_i of G by the complete bipartite graph $K_{2,m}$ for some positive integer m in such a way that the ends of e_i are merged with the two vertices part of $K_{2,m}$ after removing the edge e_i from $G.H$ is denoted by $SS(G)$.*

Definition 2.3([9]) *A supersubdivision H of a graph G is said to be an arbitrary supersubdivision of the graph G if every edge of G is replaced by an arbitrary $K_{2,m}$ (m may vary for each edge arbitrarily). H is denoted by $ASS(G)$.*

Definition 2.4 *A graph G is said to be connected if any two vertices of G are joined by a path. Otherwise it is called disconnected graph.*

Definition 2.5 *A star S_n with n spokes is given by (V, E) where $V(S_n) = \{v_0, v_1, \ldots, v_n\}$ and $E(S_n) = \{v_0v_i/i = 1, 2 \ldots, n\}$. v_0 is called the centre of the star.*

Definition 2.6 *A cycle C_n with n points is a graph given by (V, E) where $V(C_n) = \{v_1, v_2, \ldots, v_n\}$ and $E(C_n) = \{v_1v_2, v_2v_3, \ldots, v_{n-1}v_n, v_nv_1\}$.*

Theorem 2.7 *Arbitrary supersubdivisions of paths are one modulo N graceful for every positive integer N.*

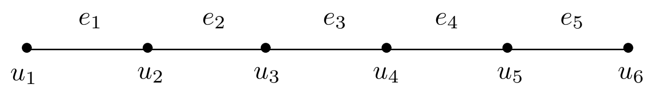
Proof Let P_n be a path with successive vertices $u_1, u_2, u_3, \cdots, u_n$ and let e_i $(1 \leq i \leq n-1)$ denote the edge u_iu_{i+1} of P_n. Let H be an arbitrary supersubdivision of the path P_n where each edge e_i of P_n is replaced by a complete bipartite graph K_{2,m_i} where m_i is any positive integer,such as those shown in Fig.1 for P_6. We observe that H has $M = 2(m_1+m_2+\cdots+m_{n-1})$ edges.

Define $\phi(u_i) = N(i-1), i = 1, 2, 3, \cdots, n$. For $k = 1, 2, 3, \cdots, m_i$, let

$$\phi(u_{i,i+1}^{(k)}) = \begin{cases} N(M-2k+1)+1 & \text{if } i = 1, \\ N(M-2k+i) - 2N(m_1+m_2+\cdots+m_{i-1})+1 & \text{if } i = 2, 3, \cdots n-1. \end{cases}$$

It is clear from the above labelling that the m_i+2 vertices of K_{2,m_i} have distinct labels and the $2m_i$ edges of K_{2,m_i} also have distinct labels for $1 \leq i \leq n-1$. Therefore, the vertices of each K_{2,m_i}, $1 \leq i \leq n-1$ in the arbitrary supersubdivision H of P_n have distinct labels and also the edges of each K_{2,m_i}, $1 \leq i \leq n-1$ in the arbitrary supersubdivision graph H of P_n have distinct labels. Also the function ϕ from the vertex set of G to $\{0, 1, N, (N+1), 2N, (2N+1), \cdots, N(q-1), N(q-1)+1\}$ is in such a way that (i) ϕ is $1-1$, and (ii) ϕ induces a bijection ϕ^* from the edge set of G to $\{1, N+1, 2N+1, \cdots, N(q-1)+1\}$, where $\phi^*(uv) = |\phi(u)-\phi(v)|$.

Hence H is one modulo N graceful.



Path P_6

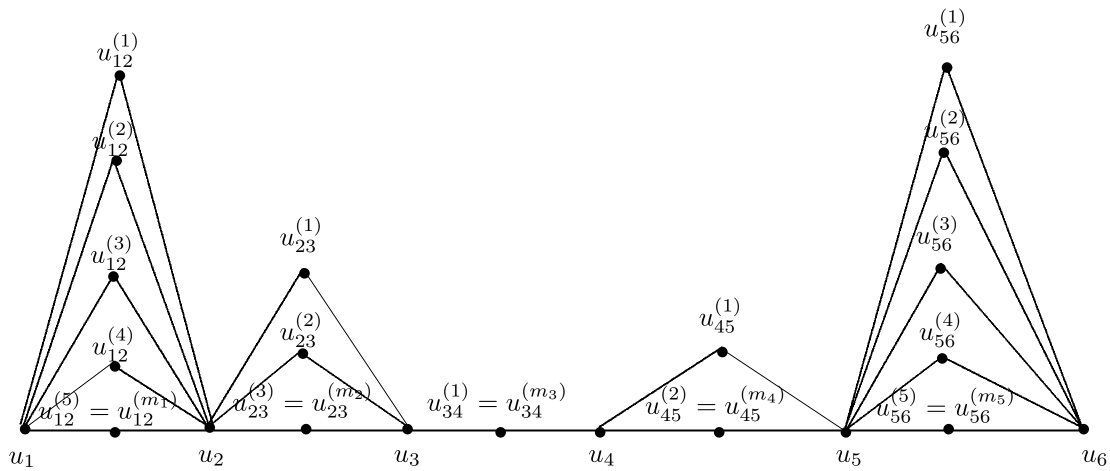


Fig.1 An arbitrary supersubdivision of P_6

Clearly, ϕ defines a one modulo N graceful labelling of arbitrary supersubdivision of the path P_n. □

Example 2.8 An odd graceful labelling of $ASS(P_5)$ is shown in Fig.2.

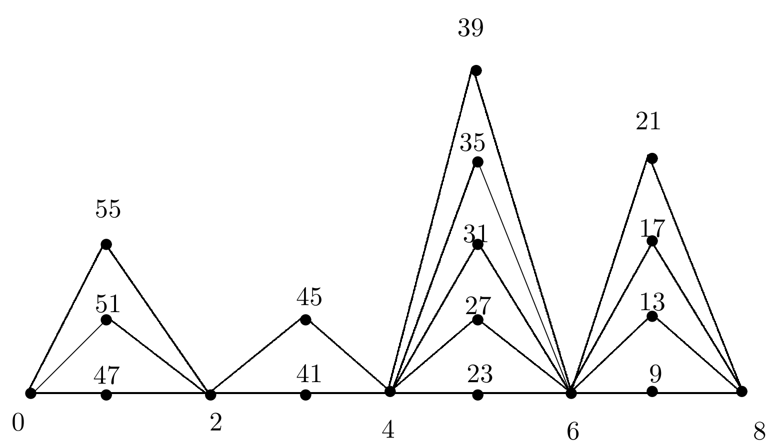


Fig.2

Example 2.9 A graceful labelling of $ASS(P_6)$ is shown in Fig.3.

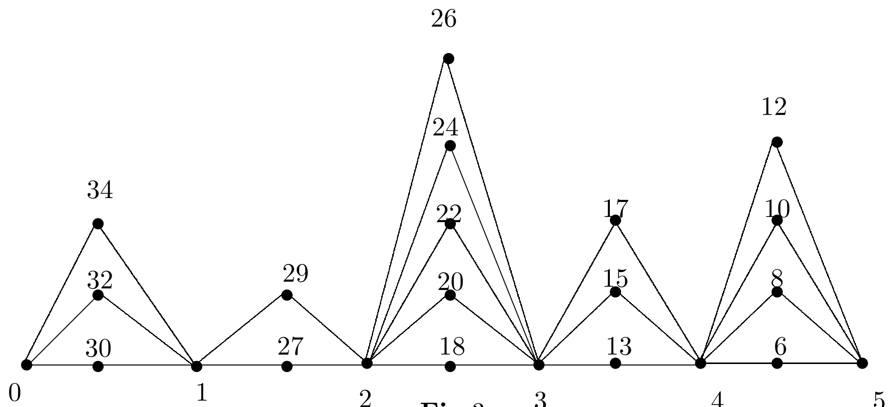


Fig.3

Example 2.10 A one modulo 7 graceful labelling of $ASS(P_6)$ is shown in Fig.4.

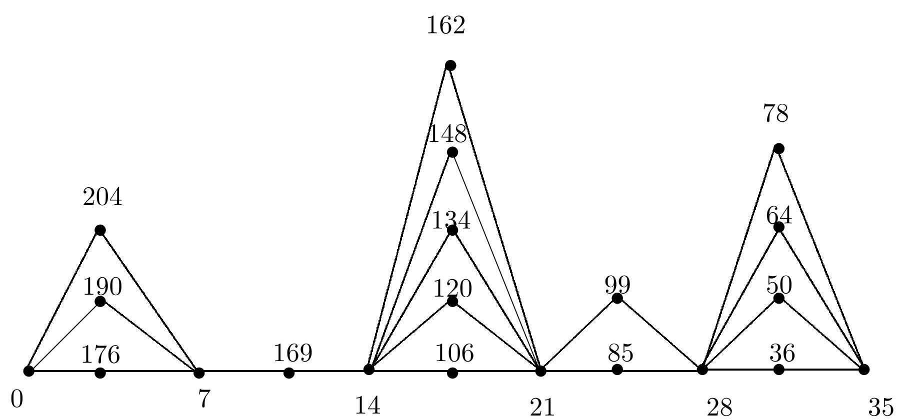


Fig.4

Theorem 2.11 *Arbitrary supersubdivision of disconnecte paths $P_n \cup P_r$ are one modulo N graceful provided the arbitrary supersubdivision is obtained by replacing each edge of G by $K_{2,m}$ with $m \geqslant 2$.*

Proof Let P_n be a path with successive vertices $v_1, v_2, \cdots, v_n$ and let e_i $(1 \leq i \leq n-1)$ denote the edge v_iv_{i+1} of P_n. Let P_r be a path with successive vertices $v_{n+1}, v_{n+2}, \cdots, v_{n+r}$ and let $e_i(n+1 \leq i \leq n+r-1)$ denote the edge v_iv_{i+1}.

Let H be an arbitrary supersubdivision of the disconnected graph $P_n \cup P_r$ where each edge e_i of $P_n \cup P_r$ is replaced by a complete bipartite graph K_{2,m_i} with $m_i \geqslant 2$ for $1 \leq i \leq n-1$ and $n+1 \leq i \leq n+r-1$. We observe that H has $M = 2(m_1+m_2+\cdots+m_{n-1}+m_{n+1}+\cdots+m_{n+r-1})$ edges.

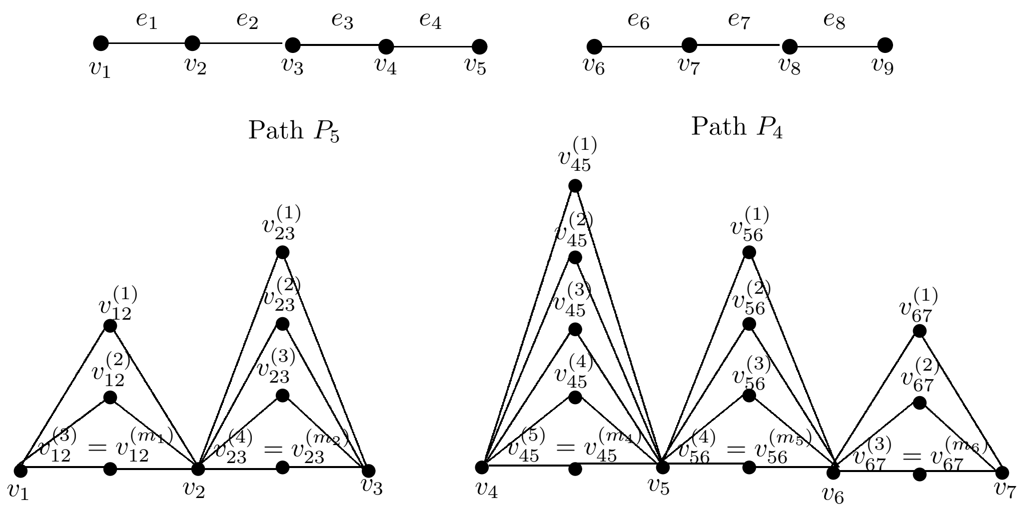


Fig.5 An arbitrary supersubdivision of $P_3 \cup P_4$

Define $\phi(v_i) = N(i-1), i = 1, 2, 3, \cdots, n$, $\phi(v_i) = N(i), i = n+1, n+2, n+3, \cdots, n+r$. For $k = 1, 2, 3, \ldots, m_i$, let

$$\phi(v_{i,i+1}^{(k)}) = \begin{cases} N(M-2k+1)+1 \quad \text{if } i = 1, \\ N(M-2+i)+1-2N(m_1+m_2+\cdots+m_{i-1}+k-1) \quad \text{if } i = 2, 3, \cdots n-1, \\ N(M-1+i)+1-2N(m_1+m_2+\cdots+m_{n-1}+k-1) \quad \text{if } i = n+1, \\ N(M-1+i)+1-2N[(m_1+m_2+\cdots+m_{n-1})+ \\ (m_{n+1}+\cdots+m_{i-1})+k-1] \quad \text{if } i = n+2, n+3, \cdots n+r-1. \end{cases}$$

It is clear from the above labelling that the m_i+2 vertices of K_{2,m_i} have distinct labels and the $2m_i$ edges of K_{2,m_i} also have distinct labels for $1 \leq i \leq n-1$ and $n+1 \leq i \leq n+r-1$.Therefore the vertices of each K_{2,m_i}, $1 \leq i \leq n-1$ and $n+1 \leq i \leq n+r-1$ in the arbitrary supersubdivision H of $P_n \cup P_r$ have distinct labels and also the edges of each K_{2,m_i}, $1 \leq i \leq n-1$ and $n+1 \leq i \leq n+r-1$ in the arbitrary supersubdivision graph H of $P_n \cup P_r$ have distinct labels. Also the function ϕ from the vertex set of G to $\{0, 1, N, (N+1), 2N, (2N+1), \ldots, N(q-1), N(q-1)+1\}$ is in such a way that (i) ϕ is $1-1$, and (ii) ϕ induces a bijection ϕ^* from the edge set of G to $\{1, N+1, 2N+1, \cdots, N(q-1)+1\}$, where $\phi^*(uv)$=$|\phi(u)-\phi(v)|$. Hence H is one modulo N graceful.

Clearly, ϕ defines a one modulo N graceful labelling of arbitrary supersubdivisions of disconnected paths $P_n \cup P_r$. □

Example 2.12 An odd graceful labelling of $ASS(P_6 \cup P_3)$ is shown in Fig.6.

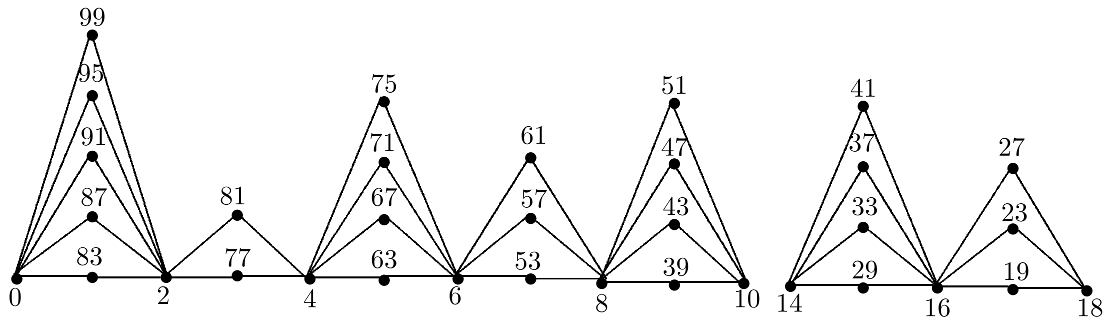


Fig.6

Example 2.13 A graceful labelling of $ASS(P_3 \cup P_4)$ is shown in Fig.7.

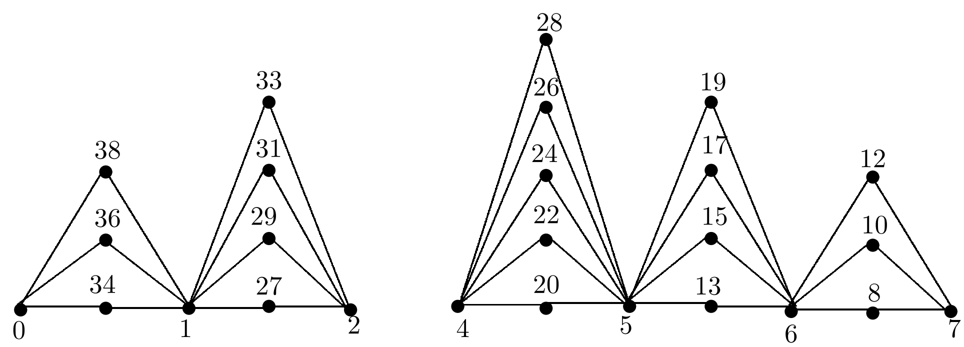


Fig.7

Example 2.14 A one modulo 4 graceful labelling of $ASS(P_4 \cup P_3)$ is shown in Fig.8.

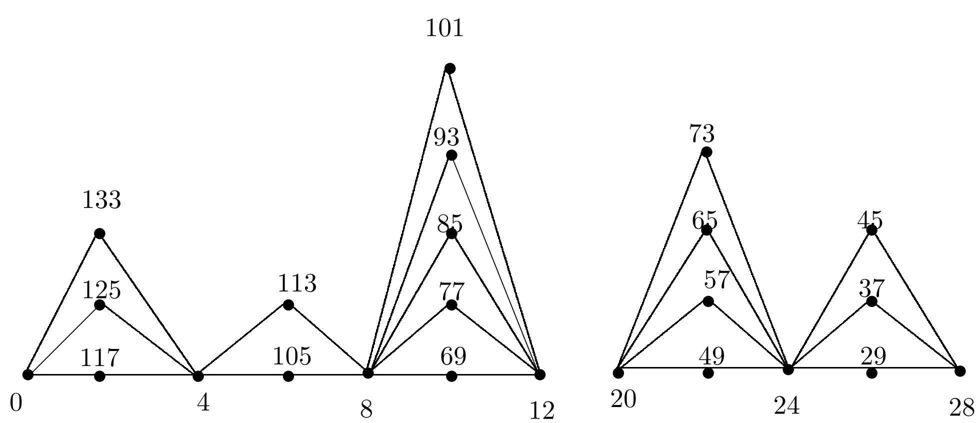


Fig.8

Theorem 2.15 *For any any $n \geq 3$, there exists an arbitrary supersubdivision of C_n which is* *one modulo N graceful for every positive integer N.*

Proof Let C_n be a cycle with consecutive vertices $v_1, v_2, v_3, \cdots, v_n$. Let G be a supersubdivision of a cycle C_n where each edge e_i of C_n is replaced by a complete bipartite graph K_{2,m_i} where m_i is any positive integer for $1 \leq i \leq n-1$ and $m_n = (n-1)$. It is clear that G has $M = 2(m_1 + m_2 + \cdots + m_n)$ edges. Here the edge $v_{n-1}v_1$ is replaced by $K_{2,n-1}$ for the construction of arbitrary supersubdivision of C_n.

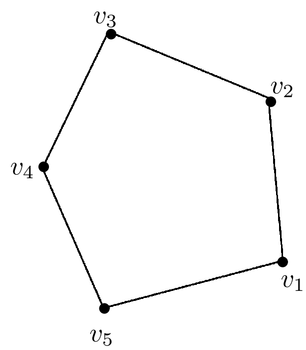


Fig.9 Cycle C_n

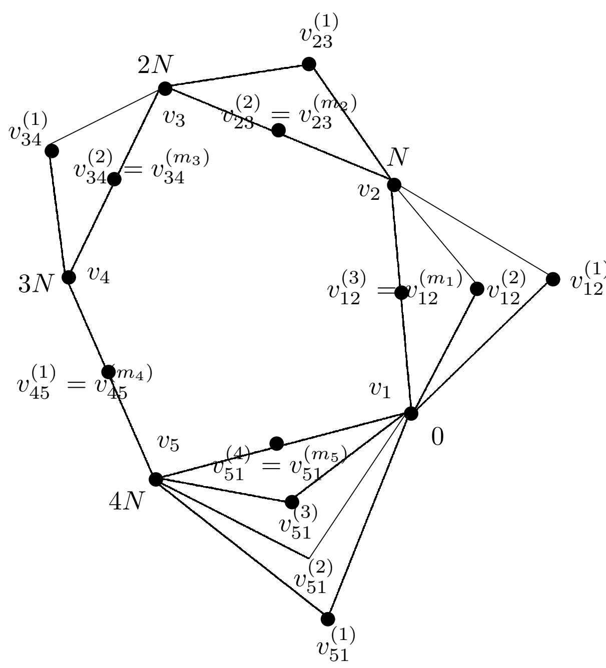


Fig.10 An arbitrary Supersubdivision of C_5

Define $\phi(v_i) = N(i-1), i = 1, 2, 3, \cdots, n$. For $k = 1, 2, 3, \ldots, m_i$, let

$$\phi(v_{i,i+1}^{(k)}) = \begin{cases} N(M-2k+1)+1 & \text{if } i = 1, \\ N(M-2k+i)+1-2N(m_1+m_2+\cdots+m_{i-1}) & \text{if } i = 2, 3, \cdots n-1. \end{cases}$$

and $\phi(v_{n,1}^{(k)}) = N(n-k+m_n-1)+1$.

It is clear from the above labelling that the function ϕ from the vertex set of G to $\{0, 1, N, (N+1), 2N, (2N+1), \cdots, N(q-1), N(q-1)+1\}$ is in such a way that (i) ϕ is $1-1$ (ii) ϕ induces a bijection ϕ^* from the edge set of G to $\{1, N+1, 2N+1, \cdots, N(q-1)+1\}$where $\phi^*(uv)$=$|\phi(u)-\phi(v)|$. Hence, H is one modulo N graceful. Clearly, ϕ defines a modulo N graceful labelling of arbitrary supersubdivision of cycle C_n. □

Example 2.16 An odd graceful labelling of $ASS(C_5)$ is shown in Fig.11.

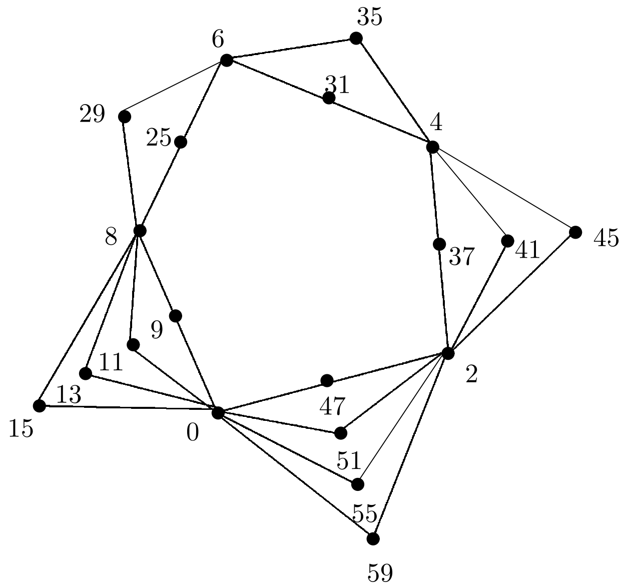


Fig.11

Example 2.17 A graceful labelling of $ASS(C_5)$ is shown in Fig.12.

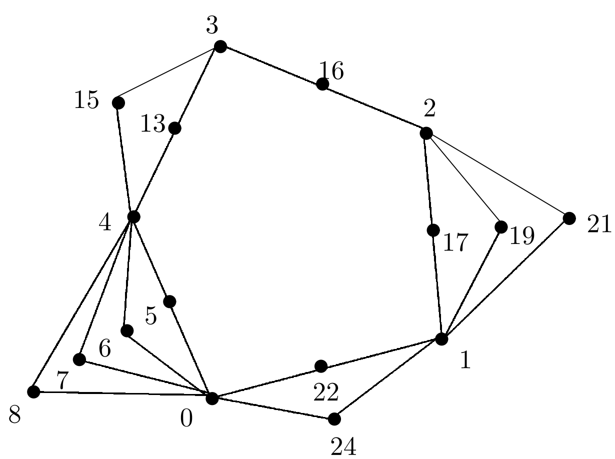


Fig.12

Example 2.18 A one modulo 3 graceful labelling of $ASS(C_4)$ is shown in Fig.13.

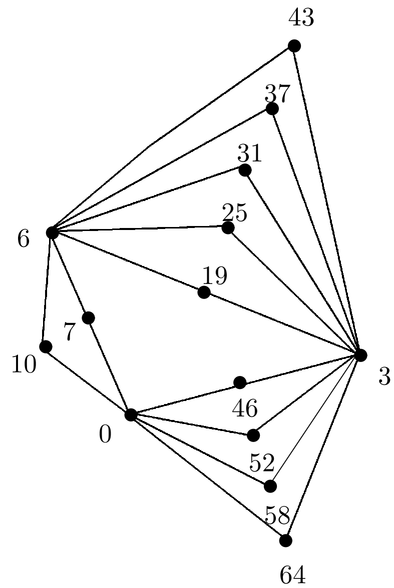


Fig.13

Theorem 2.19 *Arbitrary supersubdivision of any star is one modulo N graceful for every positive integer N.*

Proof The proof is divided into 2 cases.

Case 1 $N = 1$

It has been proved in [4] that arbitrary supersubdivision of any star is graceful.

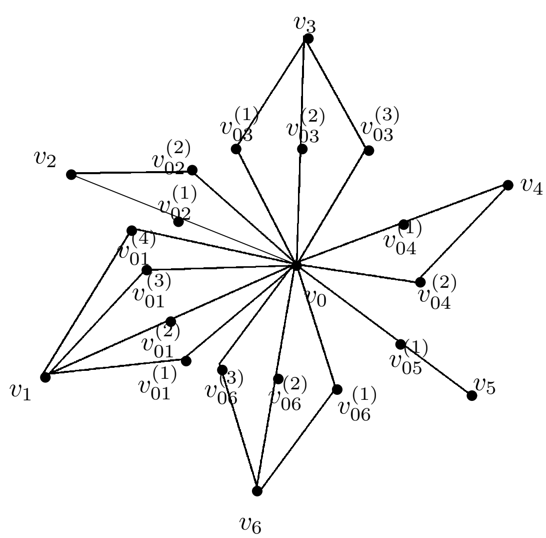


Fig.14 An arbitrary supersubdivision of S_6

Case 2 $N > 1$.

Let S_n be a star with vertices $v_0, v_1, v_2, \cdots, v_n$ and let e_i denote the edge v_0v_i of S_n for $1 \leq$

$i \leq n$. Let H be an arbitrary supersubdivision of S_n. That is for $1 \leq i \leq n$ each edge e_i of S_n is replaced by a complete bipartite graph K_{2,m_i} with m_i is any positive integer for $1 \leq i \leq n-1$ and $m_n = (n-1)$. It is clear that H has $M = 2(m_1+m_2+\cdots+m_n)$ edges. The vertex set and edge set of H are given by $V(H) = \{v_0, v_1, v_2 \cdots, v_n, v_{01}^{(1)}, v_{01}^{(2)} \cdots, v_{01}^{(m_1)}, v_{02}^{(1)}, v_{02}^{(2)}, \cdots, v_{02}^{(m_2)}, \cdots, v_{0n}^{(1)}, v_{0n}^{(2)}, \cdots, v_{0n}^{(m_n)}\}$.

Define $\phi : V(H) \rightarrow \{0, 1, 2, \cdots 2\sum_{i=1}^{n} m_i\}$ as follows:

let $\phi(v_0) = 0$. For $k = 1, 2, 3, \ldots, m_i$, let

$$\phi(v_{0i}^{(k)}) = \begin{cases} N(M-k)+1 & \text{if } i = 1, \\ N(M-k)+1-N(m_1+m_2+\cdots+m_{i-1}) & \text{if} i = 2, 3, \cdots n. \end{cases}$$

$$\phi(v_i) = \begin{cases} N(M-m_1) & \text{if } i = 1, \\ NM - N(2m_1+2m_2+\cdots+2m_{i-1}+m_i) & \text{if } i = 2, 3, \cdots n. \end{cases}$$

It is clear from the above labelling that the function ϕ from the vertex set of G to $\{0, 1, N, (N+1), 2N, (2N+1), \cdots, N(q-1), N(q-1)+1\}$ is in such a way that (i) ϕ is $1-1$ (ii) ϕ induces a bijection ϕ^* from the edge set of G to $\{1, N+1, 2N+1, \ldots, N(q-1)+1\}$where $\phi^*(uv)$=$|\phi(u) - \phi(v)|$. Hence H is one modulo N graceful.

Clearly, ϕ defines a one modulo N graceful labelling of arbitrary supersubdivision of star S_n. □

Example 2.20 A one modulo 5 graceful labelling of $ASS(S_4)$ is shown in Fig.14.

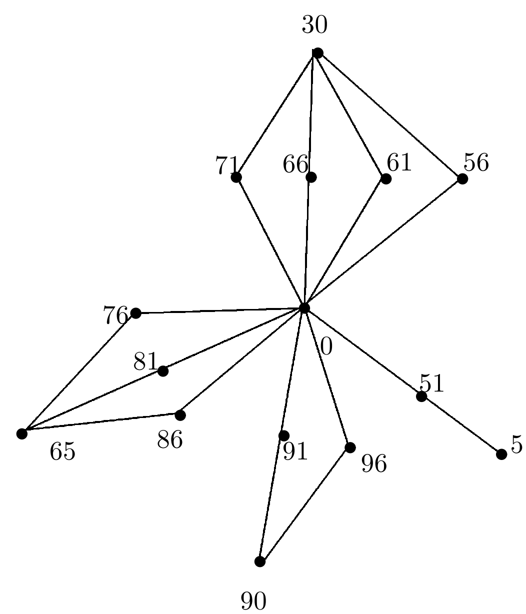


Fig.14

Example 2.21 An odd graceful labelling of $ASS(S_6)$ is shown in Fig.15.

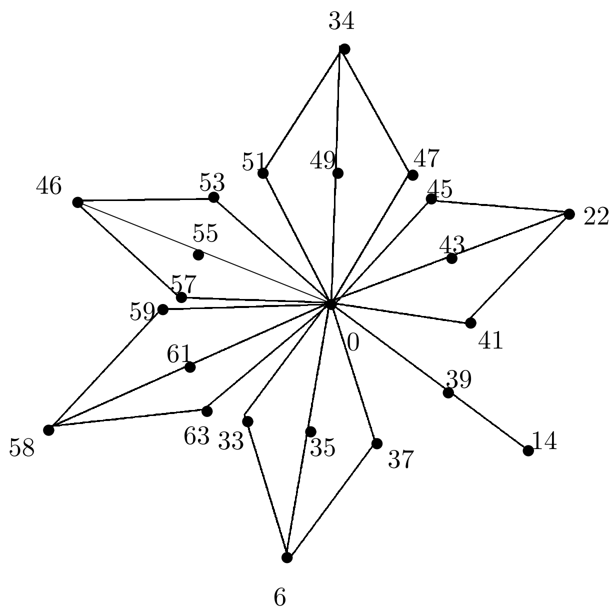


Fig.15

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