Smarandache's function applied to perfect numbers

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Smarandache's function may be defined as follows:
$S(n) =$ the smallest positive integer such that $S(n)!$ is divisible by n. [1]
In this article we are going to see that the value this function takes when n is a perfect number of the form $n = 2^{k-1} \cdot (2^k - 1)$, $p = 2^k - 1$ being a prime number.

Lemma 1 *Let $n = 2^i \cdot p$ when p is an odd prime number and i an integer such that:*

$$0 \leq i \leq E(\frac{p}{2}) + E(\frac{p}{2^2}) + E(\frac{p}{2^3}) + \cdots + E(\frac{p}{2^{E(\log_2 p)}}) = e_2(p!)$$

Where $e_2(p!)$ is the exponent of 2 in the prime number descomposition of $p!$.
$E(x)$ is the greatest integer less than or equal to x.
One has that $S(n) = p$.

Demonstration:

Given that $gcd(2^i, p) = 1$ (gcd = greatest common divisor) one has that $S(n) = max\{s(2^i), S(p)\} \geq S(p) = p$. Therefore $S(n) \geq p$.

If we prove that p! is divisible by n then one would have the equality.

$$p! = p_1^{e_{p_1}(p!)} \cdot p_2^{e_{p_2}(p!)} \cdots p_s^{e_{p_s}(p!)}$$

where p_i is the i-th prime of the prime number descomposition of $p!$. It is clear that $p_1 = 2$, $p_s = p$, $e_{p_s}(p!) = 1$ for which:

$$p! = 2^{e_2(p!)} \cdot p_2^{e_{p_2}(p!)} \cdots p_{s-1}^{e_{p_{s-1}}(p!)} \cdot p$$

From where one can deduce that:

$$\frac{p!}{n} = 2^{e_2(p!)-i} \cdot p_2^{e_{p_2}(p!)} \cdots p_{s-1}^{e_{p_{s-1}}(p!)}$$

is a positive integer since $e_2(p!) - i \geq 0$.

Therefore one has that $S(n) = p$

Proposition 1 *If n a perfec number of the form $n = 2^{k-1} \cdot (2^k - 1)$ with k a positive integer, $2^k - 1 = p$ prime, one has that $S(n) = p$.*

Demonstration:

For the Lemma it is sufficient to prove that $k - 1 \leq e_2(p!)$.

If we can prove that

$$k - 1 \leq 2^{k-1} - \frac{1}{2} \qquad (1)$$

we will have proof of the proposition since:

$$k - 1 \leq 2^{k-1} - \frac{1}{2} = \frac{2^k - 1}{2} = \frac{p}{2}$$

As k-1 is an integer one has that $k - 1 \leq E(\frac{p}{2}) \leq e_2(p!)$

Proving (1) is the same as proving $k \leq 2^{k-1} + \frac{1}{2}$ at the same time, since k is integer, is equivalent to proving $k \leq 2^{k-1}$ (2).

In order to prove (2) we may consider the function: $f(x) = 2^{x-1} - x \quad x \in \mathbb{R}$. This function may be derived and its derivate is $f'(x) = 2^{x-1} \ln 2 - 1$.

f will be increasing when $2^{x-1} \ln 2 - 1 > 0$ resolving x:

$$x > 1 - \frac{\ln(\ln 2)}{\ln 2} \simeq 1'5287$$

In particular f will be increasing $\forall\, x \geq 2$.

Therefore $\forall\, x \geq 2 \quad f(x) \geq f(2) = 0$ that is to say $2^{x-1} - x \geq 0 \quad \forall\, x \geq 2$ therefore

$$2^{k-1} \geq k \quad \forall\, k \geq 2 \quad integer$$

and thus is proved the proposition.

EXAMPLES:

$6 = 2 \cdot 3$	$S(6) = 3$
$28 = 2^2 \cdot 7$	$S(28) = 7$
$496 = 2^4 \cdot 31$	$S(496) = 31$
$8128 = 2^6 \cdot 127$	$S(8128) = 127$

References:

[1] C. Dumitrescu and R. Müller: *To Enjoy is a Permanent Component of Mathematics*. SMARANDACHE NOTIONS JOURNAL Vol. 9, No. 1-2,(1998) pp 21-26.

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