Products of Factorials in Smarandache Type Expressions

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Introduction

In [3] and [5] the authors ask how many primes are of the form $x^y + y^x$, where gcd $(x,\ y) = 1$ and $x,\ y \geq 2$. Moreover, Jose Castillo (see [2]) asks how many primes are of the Smarandache form $x_1^{x_2} + x_2^{x_3} + \ldots + x_n^{x_1}$, where $n > 1$, $x_1,\ x_2,\ \ldots,\ x_n > 1$ and gcd $(x_1,\ x_2,\ \ldots,\ x_n) = 1$ (see [9]).

In this article we announce a lower bound for the size of the largest prime divisor of an expression of the type $ax^y + by^x$, where $ab \neq 0$, $x,\ y \geq 2$ and gcd $(x,\ y) = 1$.

For any finite extension $\mathbf{F}$ of $\mathbf{Q}$ let $d_{\mathbf{F}} = [\mathbf{F} : \mathbf{Q}]$. For any algebraic number $\zeta \in \mathbf{F}$ let $N_{\mathbf{F}}(\zeta)$ denote the norm of ζ.

For any rational integer n let $P(n)$ be the largest prime number P dividing n with the convention that $P(0) = P(\pm 1) = 1$.

Theorem 1. *Let α and β be algebraic integers with $\alpha \cdot \beta \neq 0$. Let $\mathrm{K} = \mathrm{Q}[\alpha,\ \beta]$. For any two positive integers x and y let $X = \max\ (x,\ y)$. There exist computable positive numbers C_1 and C_2 depending only on α and β such that*

$$P\Big(N_{\mathrm{K}}\big(\alpha x^y + \beta y^x\big)\Big) > C_1 \left(\frac{X}{\log^3 X}\right)^{1/(d_{\mathrm{K}}+1)}$$

whenever $x,\ y \geq 2$, gcd $(x,\ y) = 1$, and $X > C_2$.

The proof of Theorem 1 uses lower bounds for linear forms in logarithms of algebraic numbers (see [1] and [7]) as well as an idea of Stewart (see [10]).

Erdös and Obláth (see [4]) found all the solutions of the equation $n! = x^p \pm y^p$ with gcd $(x,\ y) = 1$ and $p > 2$. Moreover, the author (see [6]) showed that in every non-degenerate binary recurrence sequence $(u_n)_{n \geq 0}$ there are only finitely many terms which are products of factorials.

We use Theorem 1 to show that for any two given integers a and b with $ab \neq 0$, there exist only finitely many numbers of the type $ax^y + by^x$, where $x,\ y \geq 2$ and gcd $(x,\ y) = 1$, which are products of factorials.

Let $\mathcal{PF}$ be the set of all positive integers which can be written as products of factorials; that is

$$\mathcal{PF} = \{w \mid w = \prod_{j=1}^{k} m_j!, \text{ for some } m_j \geq 1\}.$$

Theorem 2. *Let $f_1, \ldots, f_s \in Z[X, Y]$ be $s \geq 1$ homogeneous polynomials of positive degrees. Assume that $f_i(0, Y) \cdot f_i(X, 0) \not\equiv 0$ for $i = 1, \ldots, s$. Then, the equation*

$$f_1(x_1^{y_1}, y_1^{x_1}) \cdot \ldots \cdot f_s(x_s^{y_s}, y_s^{x_s}) \in \mathcal{PF}, \tag{1}$$

with $\gcd(x_i, y_i) = 1$ *and $x_i, y_i \geq 2$, for $i = 1, \ldots, s$, has finitely many solutions $x_1, y_1, \ldots, x_s, y_s$. Moreover, there exists a computable positive number C depending only on the polynomials $f_1, \ldots, f_s$ such that all solutions of equation* (1) *satisfy* $\max(x_1, y_1, \ldots, x_s, y_s) < C$.

We also have the following inhomogeneous variant of theorem 2.

Theorem 3. *Let $f_1, \ldots, f_s \in Z[X]$ be $s \geq 1$ polynomials of positive degrees. Assume that $f_i(0) \equiv 1 \pmod 2$ for $i = 1, \ldots, s$. Let $a_1, \ldots, a_s$ and $b_1, \ldots, b_s$ be $2s$ odd integers. Then, the equation*

$$f_1(a_1 x_1^{y_1} + b_1 y_1^{x_1}) \cdot \ldots \cdot f_s(a_s x_s^{y_s} + b_s y_s^{x_s}) \in \mathcal{PF}, \tag{2}$$

with $\gcd(x_i, y_i) = 1$ *and $x_i, y_i \geq 2$, for $i = 1, \ldots, s$, has finitely many solutions $x_1, y_1, \ldots, x_s, y_s$. Moreover, there exists a computable positive number C depending only on the polynomials $f_1, \ldots, f_s$ and the $2s$ numbers $a_1, b_1, \ldots, a_s, b_s$, such that all solutions of equation* (2) *satisfy* $\max(x_1, y_1, \ldots, x_s, y_s) < C$.

We conclude with the following computational results:

Theorem 4. *All solutions of the equation*

$$x^y \pm y^x \in \mathcal{PF} \qquad \text{with } \gcd(x, y) = 1 \text{ and } x, y \geq 2,$$

satisfy $\max(x, y) < \exp 177$.

Theorem 5. *All solutions of the equation*

$$x^y + y^z + z^x = n! \qquad \text{with } \gcd(x, y, z) = 1 \text{ and } x, y, z \geq 2,$$

satisfy $\max(x, y, z) < \exp 518$.

2. Preliminary Results

The proofs of theorems 1-5 use estimations of linear forms in logarithms of algebraic numbers.

Suppose that $\zeta_1, \ldots, \zeta_l$ are algebraic numbers, not 0 or 1, of heights not exceeding $A_1, \ldots, A_l$, respectively. We assume $A_m \geq e^e$ for $m = 1, \ldots, l$. Put $\Omega = \log A_1 \ldots \log A_l$. Let $\mathbf{F} = Q[\zeta_1, \ldots, \zeta_l]$. Let $n_1, \ldots, n_l$ be integers, not all 0, and let $B \geq \max |n_m|$. We assume $B \geq e^2$. The following result is due to Baker and Wüstholz.

Theorem BW ([1]). *If* $\zeta_1^{n_1}...\zeta_m^{n_l} \neq 1$, *then*

$$|\zeta_1^{n_1}...\zeta_l^{n_l} - 1| > \frac{1}{2}\exp(-(16(l+1)d_{\mathrm{F}})^{2(l+3)}\Omega \log B). \qquad (3)$$

In fact, Baker and Würtholz showed that if $\log \zeta_1$, ..., $\log \zeta_l$ are any fixed values of the logarithms, and $\Lambda = n_1 \log \zeta_1 + ... + n_l \log \zeta_l \neq 0$, then

$$\log |\Lambda| > -(16 l d_{\mathrm{F}})^{2(l+2)}\Omega \log B. \qquad (4)$$

Now (4) follows easily from (3) via an argument similar to the one used by Shorey *et al.* in their paper [8].

We also need the following p-adic analogue of theorem BW which is due to van der Poorten.

Theorem vdP ([7]). *Let* π *be a prime ideal of* F *lying above a prime integer* p. *Then,*

$$\mathrm{ord}_\pi\left(\zeta_1^{n_1}...\zeta_l^{n_l} - 1\right) < \left(16(l+1)d_{\mathrm{F}}\right)^{12(l+1)} \frac{p^{d_{\mathrm{F}}}}{\log p}\Omega\left(\log B\right)^2. \qquad (5)$$

The following estimations are useful in what follows.

Lemma 1. *Let* $n \geq 2$ *be an integer, and let* $p \leq n$ *be a prime number. Then*

(i)

$$n^{n/2} \leq n! \leq n^n. \qquad (6)$$

(ii)

$$\frac{n}{4(p-1)} \leq \mathrm{ord}_p n! \leq \frac{n}{p-1}. \qquad (7)$$

Proof. See [6].

Lemma 2. *(1) Let* $s \geq 1$ *be a positive integer. Let* C *and* X *be two positive numbers such that* $C > \exp s$ *and* $X > 1$. *Let* $y > 0$ *be such that* $y < C \log^s X$. *Then,* $y \log y < (C \log C) \log^{s+1} X$.

(2) Let $s \geq 1$ *be a positive integer, and let* $C > \exp(s(s+1))$. *If* X *is a positive number such that* $X < C \log^s X$, *then* $X < C \log^{s+1} C$.

Proof. *(1)* Clearly,

$$y \log y < C \log^s X\left(\log C + s \log\log X\right).$$

It suffices to show that

$$\log C + s \log \log X < \log C \log X.$$

The above inequality is equivalent to

$$\log C(\log X - 1) > s \log\log X.$$

This last inequality is obviously satisfied since $\log C > s$ and $\log X > \log\log X + 1$, for all $X > 1$.

(2) Suppose that $X \geq C \log^{s+1} C$. Since $s \geq 1$ and $C > \exp(s(s+1))$, it follows that $C \log^{s+1} C > C > \exp s$. The function $\frac{y}{\log^s y}$ is increasing for $y > \exp s$. Hence, since $X \geq C \log^{s+1} C$, we conclude that

$$\frac{C \log^{s+1} C}{\log^s (C \log^{s+1} C)} \leq \frac{X}{\log^s X} < C.$$

The above inequality is equivalent to

$$\frac{\log^{s+1} C}{\left(\log C + (s+1)\log\log C\right)^s} < 1,$$

or

$$\log C < \left(1 + (s+1)\frac{\log\log C}{\log C}\right)^s.$$

By taking logarithms in this last inequality we obtain

$$\log\log C < s \log\left(1 + (s+1)\frac{\log\log C}{\log C}\right) < s(s+1)\frac{\log\log C}{\log C}.$$

This last inequality is equivalent to $\log C < s(s+1)$, which contradicts the fact that $C > \exp(s(s+1))$.

3. The Proofs

The Proof of Theorem 1. By C_1, C_2, ..., we shall denote computable positive numbers depending only on the numbers α and β. Let $d = d_{\mathbf{K}}$. Let

$$N_{\mathbf{K}}(\alpha x^y + \beta y^x) = p_1^{\delta_1} \cdot \ldots \cdot p_k^{\delta_k}$$

where $2 < p_1 < p_2 < \ldots < p_k$ are prime numbers. For $\mu = 1, \ldots, d$, let $\alpha^{(\mu)} x^y + \beta^{(\mu)} y^x$ be a conjugate, in K, of $\alpha x^y + \beta y^x$. Fix $i = 1, \ldots, k$. Let π be a prime ideal of K lying above p_i. We use theorem vdP to bound $\mathrm{ord}_\pi(\alpha^{(\mu)} x^y + \beta^{(\mu)} y^x)$. We distinguish two cases:

CASE 1. $p_i \mid xy$. Suppose, for example, that $p_i \mid y$. Since $(x, y) = 1$, it follows that $p_i \nmid x$. Hence, by theorem vdP,

$$\mathrm{ord}_\pi(\alpha^{(\mu)} x^y + \beta^{(\mu)} y^x) = \mathrm{ord}_\pi(\alpha^{(\mu)} x^y) + \mathrm{ord}_\pi\left(1 - \left(-\frac{\beta^{(\mu)}}{\alpha^{(\mu)}}\right) y^x x^{-y}\right) <$$

$$< C_1 + C_2 \frac{p_i^d}{\log p_i} \log^4 X. \tag{8}$$

where $C_1 = d \cdot \log_2 N_{\mathrm{K}}(\alpha)$, and C_2 can be computed in terms of α and β using theorem vdP.

CASE 2. $p_i \not| \, xy$. In this case

$$\mathrm{ord}_\pi\left(\alpha^{(\mu)} x^y + \beta^{(\mu)} y^x\right) = \mathrm{ord}_\pi\left(\alpha^{(\mu)} x^y\right) + \mathrm{ord}_\pi\left(1 - \left(-\frac{\beta^{(\mu)}}{\alpha^{(\mu)}}\right) \cdot \frac{y^x}{x^y}\right) <$$

$$< C_1 + C_2 \frac{p_i^d}{\log p_i} \log^4 X. \tag{9}$$

Combining Case 1 and Case 2 we conclude that

$$\mathrm{ord}_\pi\left(\alpha^{(\mu)} x^y + \beta^{(\mu)} y^x\right) < C_3 \frac{p_i^d}{\log p_i} \log^4 X, \tag{10}$$

where $C_3 = 2 \cdot \max\,(C_1,\ C_2)$. Hence,

$$\delta_i = \mathrm{ord}_{p_i}\left(N_{\mathrm{K}}\left(\alpha x^y + \beta y^x\right)\right) < C_4 \frac{p_i^d}{\log p_i} \log^4 X. \tag{11}$$

where $C_4 = dC_3$. Denote p_k by P. Since $p_i \le P$ for $i = 1, \ldots, k$, it follows, by formula (11), that

$$\log\left(N_{\mathrm{K}}\left(\alpha x^y + \beta y^x\right)\right) \le \sum_{i=1}^{k} \delta_i \cdot \log p_i < k C_4 P^d \log^4 X. \tag{12}$$

Clearly $k \le \pi(P)$, where $\pi(P)$ is the number of primes less than or equal to P. Combining inequality (12) with the prime number theorem we conclude that

$$\log\left(N_{\mathrm{K}}\left(\alpha x^y + \beta y^x\right)\right) < C_5 \frac{P^{d+1}}{\log P} \log^4 X. \tag{13}$$

We now use theorem BW to find a lower bound for $\log\left(N_{\mathrm{K}}\left(\alpha x^y + \beta y^x\right)\right)$. Suppose that $X = y$. For $\mu = 1, \ldots, d$, we have

$$\log\left(\left|\alpha^{(\mu)} x^y + \beta^{(\mu)} y^x\right|\right) = \log\left(\left|\alpha^{(\mu)} x^y\right|\right) + \log\left(\left|1 - \left(-\frac{\beta^{(\mu)}}{\alpha^{(\mu)}}\right) \frac{y^x}{x^y}\right|\right) >$$

$$> C_6 + X \log 2 - C_7 \log^3 X.$$

where $C_6 = \min\,\left(\log\left|\alpha^{(\mu)}\right| \mid \mu = 1, \ldots, d\right)$, and C_7 can be computed using theorem BW. Hence,

$$\log\left(N_{\mathrm{K}}\left(\alpha x^y + \beta y^x\right)\right) > dC_6 + dX \log 2 - dC_7 \log^3 X. \tag{14}$$

Let $C_8 = dC_6$, $C_9 = d\log 2$, and $C_{10} = dC_7$. Let also C_{11} be the smallest positive number such that

$$\frac{1}{2}C_9 y > C_{10}\log^3 y - C_8, \qquad \text{for } y > C_{11}.$$

Combining inequalities (13) and (14) it follows that

$$C_5 \frac{P^{d+1}}{\log P}\log^4 X > C_8 + C_9 X - C_{10}\log^3 X > \frac{1}{2}C_9 X, \tag{15}$$

for $X \geq C_{11}$. Inequality (15) clearly shows that

$$P > C_{12}\left(\frac{X}{\log^3 X}\right)^{\frac{1}{d+1}}, \qquad \text{for } X \geq C_{11}.$$

The Proof of Theorem 2. By C_1, C_2, ..., we shall denote computable positive numbers depending only on the polynomials f_1, ..., f_s. We may assume that f_1, ..., f_s are linear forms with algebraic coefficients. Let $f_i(X, Y) = \alpha_i X + \beta_i Y$ where $\alpha_i\beta_i \neq 0$, and let $\mathrm{K} = \mathrm{Q}[\alpha_1, \beta_1, \ldots, \alpha_s, \beta_s]$. Let $(x_1, y_1, \ldots, x_s, y_s)$ be a solution of (1). Equation (1) implies that

$$\prod_{i=1}^{s} N_{\mathrm{K}}(\alpha_i x_i^{y_i} + \beta_i y_i^{x_i}) = n_1! \cdot \ldots \cdot n_k! \tag{16}$$

We may assume that $2 \leq n_1 \leq n_2 \leq \ldots \leq n_k$. Let $X = \max(x_i, y_i \mid i = 1, \ldots, s)$. It follows easily, by inequality (10), that

$$\mathrm{ord}_2\left(\prod_{i=1}^{s} N_{\mathrm{K}}(\alpha_i x_i^{y_i} + \beta_i y_i^{x_i})\right) < C_1 \log^4 X. \tag{17}$$

Hence,

$$\sum_{i=1}^{k} \mathrm{ord}_2 n_i! < C_1 \log^4 X.$$

By lemma 1, it follows that

$$n_k < 4C_1 \log^4 X. \tag{18}$$

On the other hand, by theorem 1, there exists computable constants C_{2i} and C_{3i}, such that

$$P\Big(N_{\mathrm{K}}(\alpha_i x_i^{y_i} + \beta_i y_i^{x_i})\Big) > C_{2i}\left(\frac{X_i}{\log^3 X_i}\right)^{1/(d_{\mathrm{K}}+1)} \tag{19}$$

whenever $x_i,\ y_i \geq 2$, gcd $(x_i,\ y_i) = 1$ and $X_i = \max(x_i,\ y_i) > C_{3i}$. Let $C_2 = \min(C_{2i} \mid i = 1, \ldots, s)$ and let $C_3 = \max(C_{3i} \mid i = 1, \ldots, s)$. Suppose that $X > C_3$. From inequality (19) we conclude that

$$P\left(\prod_{i=1}^{s} N_{\mathrm{K}}\left(\alpha_i x_i^{y_i} + \beta_i y_i^{x_i}\right)\right) > C_2\left(\frac{X}{\log^3 X}\right)^{1/(d_{\mathrm{K}}+1)}. \tag{20}$$

Since $P \mid \prod_{i=1}^{k} n_i!$, it follows that $P \leq n_k$. Combining inequalities (18) and (20) we conclude that

$$C_2\left(\frac{X}{\log^3 X}\right)^{1/(d_{\mathrm{K}}+1)} < 4C_1 \log^4 X. \tag{21}$$

Inequality (21) clearly shows that $X < C_4$.

The Proof of Theorem 3. By $C_1,\ C_2, \ldots$, we shall denote computable positive numbers depending only on the polynomials $f_1, \ldots, f_s$ and on the numbers $a_1,\ b_1, \ldots, a_s,\ b_s$. Let $(x_1,\ y_1, \ldots, x_s,\ y_s)$ be a solution of (2). Let $X_i = \max(x_i,\ y_i)$, and let $X = \max(X_i \mid i = 1, \ldots, s)$. Finally, let

$$f_i(Z) = c_i \prod_{j=1}^{d_i} (Z - \zeta_{i,j}).$$

Let $K = \mathrm{Q}[\zeta_{i,j}]_{\substack{1\leq i\leq s \\ 1\leq j\leq d_i}}$, and let $d = [\mathrm{K} : \mathrm{Q}]$, $D = \sum_{i=1}^{s} d_i$, and $c = \prod_{i=1}^{s} c_i$. Let π be a prime ideal of K lying above 2. Let $Z_i = a_i x_i^{y_i} + b_i y_i^{x_i}$. We first bound $\mathrm{ord}_\pi f_i(Z_i)$. First, notice that $\mathrm{ord}_\pi(a_i b_i) = 0$. Moreover, since $f_i(0) \equiv 1 \pmod 2$, it follows that $\mathrm{ord}_\pi(\zeta_{i,j}) = 0$, for all $j = 1, \ldots, d_i$. We distinguish 2 cases:

CASE 1. Assume that $2 \nmid x_i y_i$. Then $f_i(Z_i) \equiv f_i(0) \equiv 1 \pmod 2$. Hence, $\mathrm{ord}_\pi f_i(Z_i) = 0$.

CASE 2. Assume that $2 \mid x_i$. In this case, $\mathrm{ord}_\pi(y) = 0$. Fix $j = 1, \ldots, d_i$. Then,

$$\mathrm{ord}_\pi\left(Z_i - \zeta_{i,j}\right) = \mathrm{ord}_\pi\Big(a_i x_i^{y_i} + \left(b_i y_i^{x_i} - \zeta_{i,j}\right)\Big). \tag{22}$$

Since $\mathrm{ord}_\pi(b_i y_i^{x_i}) = \mathrm{ord}_\pi(\zeta_{i,j}) = 0$, it follows, by theorem vdP, that

$$\mathrm{ord}_\pi\left(b_i y_i^{x_i} - \zeta_{i,j}\right) = \mathrm{ord}_\pi\Big(b_i y_i^{x_i} (\zeta_{i,j})^{-1} - 1\Big) < C_1 \log^3 X_i. \tag{23}$$

We distinguish 2 cases:

CASE 2.1. $y_i \geq C_1 \log^3 X_i$. In this case, from formula (22) and inequality (23), it follows that

$$\operatorname{ord}_\pi(Z_i - \zeta_{i,j}) = \operatorname{ord}_\pi(b_i y_i^{x_i} - \zeta_{i,j}) < C_1 \log^3 X_i. \tag{24}$$

CASE 2.2. $y_i < C_1 \log^3 X_i$. In this case,

$$\operatorname{ord}_\pi(Z_i - \zeta_{i,j}) = \operatorname{ord}_\pi\Big(b_i y_i^{x_i} + \big(a_i x_i^{y_i} - \zeta_{i,j}\big)\Big). \tag{25}$$

Let $\Delta = a_i x_i^{y_i} - \zeta_{i,j}$. Let $H(\Delta)$ be the height of Δ. Clearly,

$$H(\Delta) < C_2 x_i^{d_i y_i}.$$

Hence,

$$\log(H(\Delta)) < \log C_2 + d_i y_i \log x_i < C_3 + C_4 \log^4 X_i,$$

where $C_3 = \log C_2$, and $C_4 = C_1 \cdot \max\,(d_i \mid i = 1,\ ...,\ s)$. Since $\operatorname{ord}_\pi(b_i) = \operatorname{ord}_\pi(y_i^{x_i}) = 0$, it follows, by theorem vdP, that

$$\operatorname{ord}_\pi(Z_i - \zeta_{i,j}) = \operatorname{ord}_\pi(1 - b_i^{-1} y_i^{-x_i}\Delta) < C_5 \log y_i \log(H(\Delta)) \log^2 x_i <$$

$$< C_5 \log^3 X_i (C_3 + C_4 \log^4 X_i). \tag{26}$$

Let $C_6 = 2C_4C_5$. Also, let

$$C_7 = \exp\big((C_3/C_4)^{1/4}\big).$$

From inequalities (23) and (26), it follows that

$$\operatorname{ord}_\pi(Z_i - \zeta_{i,j})) < C_6 \log^7 X, \qquad \text{for } X > C_7. \tag{27}$$

Hence,

$$\operatorname{ord}_2\Big(\prod_{i=1}^{s} f_i(Z_i)\Big) < C_8 \log^7 X, \qquad \text{for } X > C_7, \tag{28}$$

where $C_8 = 2\max\,\big(sDC_6,\ c\big)$. Suppose now that

$$\prod_{i=1}^{s} f_i(Z_i) = \prod_{j=1}^{k} n_j!, \tag{29}$$

where $2 \leq n_1 \leq n_2 \leq ... \leq n_k$. From inequality (28) and lemma 1, it follows that

$$\sum_{j=1}^{k} n_j < C_9 \log^7 X,$$

where $C_9 = 4C_8$. Hence,

$$\log\left(\prod_{j=1}^{k} n_j!\right) = \sum_{j=1}^{k} \log n_j! < \sum_{j=1}^{k} n_j \log n_j < \left(\sum_{j=1}^{k} n_j\right) \log\left(\sum_{j=1}^{k} n_j\right) <$$

$$< C_9 \log^7 X\left(\log C_9 + 7 \log\log X\right), \qquad \text{for } X > C_7. \qquad (30)$$

Let C_{10} be the smallest positive number $\geq C_7$ such that

$$y > \log C_9 + 7 \log\log y, \qquad \text{for } y > C_{10}.$$

From inequality (30), it follows that

$$\log\left(\prod_{j=1}^{k} n_j!\right) < C_9 \log^8 X, \qquad \text{whenever } X > C_{10}. \qquad (31)$$

We now bound $\log\left(\prod_{i=1}^{s} f_i(Z_i)\right)$. Fix $i = 1, \ldots, s$. Suppose that $y_i = X_i$. By Theorem BW,

$$\log|Z_i| = \log|a_i x_i^{y_i} + b_i y_i^{x_i}| = \log\left(|a_i| x_i^{y_i}\right) + \log\left(\left|1 - \left(-\frac{b_i}{a_i}\right) y_i^{x_i} x_i^{-y_i}\right|\right) >$$

$$> C_{11} + X_i \log 2 - C_{12} \log^3 X_i, \qquad (32)$$

where $C_{11} = \min\left(|a_i| \mid i = 1, \ldots, s\right)$, and C_{12} can be computed using theorem BW. Let $C_{13} = (\log 2)/2$, and let C_{14} be the smallest positive number $\geq C_{10}$ such that

$$C_{11} + y \log 2 - C_{12} \log^3 y > C_{13} y, \qquad \text{for } y > C_{14}.$$

From inequality (32) it follows that

$$\max\left(\log|Z_i|\right) > C_{13} X, \qquad \text{for } X > C_{14}. \qquad (33)$$

On the other hand, for each $i = 1, \ldots, s$, there exists two computable constants C_i and C_i' such that

$$|f_i(Z_i)| > C_i |Z_i|^{d_i}, \qquad \text{whenever } |Z_i| > C_i'.$$

Let $C_{15} = \min\left(C_i \mid i = 1, \ldots, s\right)$, and let $C_{16} = \max\left(C_i' \mid i = 1, \ldots, s\right)$. Finally, let $C_{17} = \max\left(C_{14}, (\log C_{16})/C_{13}\right)$. Suppose that $X > C_{17}$. Since $|f_i(Z_i)| \geq 1$, for all $i = 1, \ldots, s$, it follows, by inequality (33), that

$$\log\left(\prod_{i=1}^{s} f_i(Z_i)\right) \geq \max\left(\log|f_i(Z_i)| \; i = 1, \ldots, s\right) >$$

$$> \log C_{15} + \max\big(\log|Z_i| \mid i = 1, \ldots, s\big) > \log C_{15} + C_{13}X, \quad \text{for } X > C_{17}. \tag{34}$$

From equation (29) and inequalities (31) and (34), it follows that

$$\log C_{15} + C_{13}X < C_9 \log^8 X, \qquad \text{for } X > C_{17}. \tag{35}$$

Inequality (35) clearly shows that $X < C_{18}$.

The Proof of Theorem 4. Let $X = \max(x, y)$. Notice that if $x^y \pm y^x \in \mathcal{PF}$, than xy is odd. Hence, by theorem vdP,

$$\mathrm{ord}_2(x^y \pm y^x) = \mathrm{ord}_2(1 - (\mp y)^x x^{-y}) < 48^{36} \cdot \frac{2}{\log 2} \cdot \log^4 X. \tag{36}$$

Suppose that

$$x^y \pm y^x = n_1! \cdot \ldots \cdot n_k!, \tag{37}$$

where $2 \le n_1 \le \ldots \le n_k$. From inequality (36) and lemma 1 it follows that

$$\sum_{i=1}^{k} n_i \le 4\Big(\sum_{i=1}^{k} \mathrm{ord}_2(n_i!)\Big) < 48^{36} \cdot \frac{8}{\log 2} \cdot \log^4 X < 12 \cdot 48^{36} \cdot \log^4 X. \tag{38}$$

It follows, by lemma 2 *(1)*, that

$$\log(x^y \pm y^x) = \log \prod_{i=1}^{k} n_i! = \sum_{i=1}^{k} \log n_i! < \sum_{i=1}^{k} n_i \log n_i <$$

$$< \Big(\sum_{i=1}^{k} n_i\Big) \log\Big(\sum_{i=1}^{k} n_i\Big) < 12 \cdot 48^{36} \log\big(12 \cdot 48^{36}\big) \cdot \log^5 X < 1703 \cdot 48^{36} \log^5 X. \tag{39}$$

Suppose now that $X = y$. Then, by theorem BW,

$$\log|x^y \pm y^x| \ge \log|x^y - y^x| = \log(x^y) + \log|1 - y^x x^{-y}| >$$

$$> X \log 3 - \log 2 - 48^{10} \log^3 X. \tag{40}$$

Combining inequalities (39) and (40), we conclude that

$$X < X \log 3 < \log 2 + 48^{10} \log^3 X + 1703 \cdot 48^{36} \log^5 X < 1704 \cdot 48^{36} \log^5 X. \tag{41}$$

Let $C = 1704 \cdot 48^{36}$, and let $s = 5$. Since $\log C = \log 1704 + 36 \log 48 > 30$, it follows, by lemma 2 *(2)*, that

$$X < C \cdot \log^6 C < 1704 \cdot 48^{36} \cdot 147^6. \tag{42}$$

Hence, $\log X < 177$.

The Proof of Theorem 5. Suppose that (x, y, z, n) is a solution of $x^y + y^z + z^x = n!$, with gcd $(x, y, z) = 1$ and min $(x, y, z) > 1$. Let $X = \max(x, y, z)$. We assume that $\log X > 519$. Clearly, not all three numbers x, y, z can be odd. We may assume that $2 \mid x$. In this case, both y and z are odd. By theorem vdP,

$$\text{ord}_2(y^z + z^x) = \text{ord}_2(1 - (-y)^{-z} z^x) < 48^{36} \frac{2}{\log 2} \log^4 X < 3 \cdot 48^{36} \log^4 X. \tag{43}$$

We distinguish two cases:

CASE 1. $y \geq 3 \cdot 48^{36} \log^4 X$. In this case, by lemma 1,

$$n/4 \leq \text{ord}_2 n! = \text{ord}_2(x^y + y^z + z^x) = \text{ord}_2(y^z + z^x) < 3 \cdot 48^{36} \log^4 X. \tag{44}$$

Hence,

$$n < 12 \cdot 48^{36} \log^4 X. \tag{45}$$

By lemma 2 *(1)*, it follows that

$$n \log n < 12 \cdot 48^{36} \log(12 \cdot 48^{36}) \log^5 X < 1703 \cdot 48^{36} \log^5 X. \tag{46}$$

We conclude that

$$X \log 2 < \log(x^y + y^z + z^x) = \log n! < n \log n < 1703 \cdot 48^{36} \log^5 X.$$

Let $C = 1703 \cdot 48^{36} / \log 2$, and let $s = 5$. Since $\log C > 30$, it follows, by lemma 2 *(2)*, that

$$X < C \log^6 C < 2457 \cdot 48^{36} \cdot 148^6.$$

Hence, $\log X < 178$, which is a contradiction.

CASE 2. $y < 3 \cdot 48^{36} \log^4 X$. Let p be a prime number such that $p \mid y$. We first show that $p \nmid x$. Indeed, assume that $p \mid x$. Since gcd $(x, y, z) = 1$, it follows that $p \nmid z$. QWe conclude that $p \nmid n!$, therefore $n < p$. Hence,

$$n < p \leq y < 3 \cdot 48^{36} \log^4 X.$$

In particular, n satisfies inequality (45). From Case 1 we know that $\log X < 178$, which is a contradiction.

Suppose now that $p \nmid x$. Then, by theorem vdP,

$$\text{ord}_p(x^y + z^x) = \text{ord}_p(1 - (-x)^{-y} z^x) < 48^{36} \frac{p}{\log p} \log^4 X <$$

$$< 48^{36} y \log^4 X < 3 \cdot 48^{72} \log^8 X. \tag{47}$$

We distinguish 2 cases:

CASE 2.1. $z \geq 3 \cdot 48^{72} \log^8 X$. In this case, by lemma 2 *(1)* and inequality (47),

$$\frac{n}{4(p-1)} < \mathrm{ord}_p n! = \mathrm{ord}_p\big(y^z + (x^y + z^x)\big) =$$

$$= \mathrm{ord}_p(x^y + z^x) < 3 \cdot 48^{72} \log^8 X.$$

Hence,

$$n < 12(p-1) \cdot 48^{72} \log^8 X < 12y \cdot 48^{72} \log^8 X < 36 \cdot 48^{108} \log^{12} X. \quad (48)$$

From lemma 2 *(1)* we conclude that

$$X \log 2 < \log(x^y + y^z + z^x) = \log n! < n \log n <$$

$$< 36 \cdot 48^{108} \log(36 \cdot 48^{108}) \log^{13} X < 317 \cdot 48^{109} \log^{13} X. \quad (49)$$

Let $C = 317 \cdot 48^{109} / \log 2$, and let $s = 13$. Since $\log C > 182$, it follows, by lemma 2 *(2)*, that

$$X < C \log^{11} C < 458 \cdot 48^{109} \ln^{14}(458 \cdot 48^{109}) < 458 \cdot 48^{109} \cdot 429^{14}.$$

Hence, $\log X < 513$, which is a contradiction.

CASE 2.2. $z < 3 \cdot 48^{72} \log^8 X$. By theorem vdP, it follows that

$$\mathrm{ord}_2\big(z^x + (x^y + y^z)\big) = \mathrm{ord}_2\big(1 - (-x^y - y^z) z^{-X}\big) <$$

$$< 48^{36} \frac{2}{\log 2} \log(x^y + y^z) \log^3 X < 3 \cdot 48^{36} \log(x^y + y^z) \log^3 X. \quad (50)$$

We now bound $\log(x^y + y^z)$. Let $y_1 = 3 \cdot 48^{36} \log^4 X$ and $z_1 = 3 \cdot 48^{72} \log^8 X$. Since $y < y_1$ and $z < z_1$, it follows that

$$\log(x^y + y^z) < \log(X^{y_1} + y_1^{z_1}) < \log 2 + \max\,(y_1 \log X,\ z_1 \log y_1).$$

Since $z_1 \log y_1 > z_1 > y_1 \log X$, it follows that

$$\log(x^y + y^z) < \log 2 + z_1 \log y_1.$$

From lemma 2 *(1)* we conclude that

$$\log(x^y + y^z) < \log 2 + z_1 \log y_1 = \log 2 + \frac{z_1}{y_1} \cdot (y_1 \log y_1) <$$

$$< \log 2 + 48^{36} \log^4 X \cdot \Big(3 \cdot 48^{36} \log(3 \cdot 48^{36})\Big) \log^5 X < 422 \cdot 48^{72} \log^9 X. \quad (51)$$

From lemma 1 and inequalities (50) and (51) it follows that

$$n/4 < \mathrm{ord}_2 n! = \mathrm{ord}_2\big(z^x + (x^y + y^z)\big) < 1266 \cdot 48^{108} \log^{12} X.$$

Hence,

$$n < 5064 \cdot 48^{108} \log^{12} X.$$

By lemma 2 *(1)*, it follows that

$$X \log 2 < \log(x^y + y^z + z^x) = \log n! < n \log n <$$

$$< 5064 \cdot 48^{108} \cdot \log(5064 \cdot 48^{108}) \log^{13} X < 22 \cdot 48^{111} \log^{13} X.$$

Let $C = 22 \cdot 48^{111} / \log 2$, and let $s = 13$. Since $\log C > 182$, it follows, by lemma 2 *(2)*, that

$$X < C \log^{14} C < 22 \cdot 48^{111} \cdot 433^{14}.$$

Hence, $\log X < 518$, which is the final contradiction.

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