SOME REMARKS CONCERNING THE DISTRIBUTION OF THE SMARANDACHE FUNCTION

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The Smarandache function is a numerical function $S: N^* \to N^*$ S(k) representing the smallest natural number n such that n! is divisible by k. From the definition it results that S(1)=1.

I will refer for the beginning the following problem:

"Let k be a rational number, $0 < k \le 1$. Does the diophantine equation $\frac{S(n)}{n} = k$ has always solutions? Find all k such that the equation has an infinite number of solutions in N^*" from "Smarandache Function Journal".

I intend to prove that equation hasn't always solutions and case that there are an infinite number of solutions is when $k = \frac{1}{r}$, $r \in N^*$, $k \in Q$ and $0 < k \le 1 \Rightarrow$ there are two relatively prime non negative integers p and q such that $k = \frac{q}{p}$, $p,q \in N^*$, $0 < q \le p$. Let n be a solution of the equation $\frac{S(n)}{n} = k$. Then $\frac{S(n)}{n} = \frac{p}{q}$, (1). Let d be a highest common divisor of n and S(n) : d = (n, S(n)). The fact that p and q are relatively prime and (1) implies that S(n) = qd , n = pd $\Rightarrow$ S(pd) = qd (*).

This equality gives us the following result: (qd)! is divisible by pd $\Rightarrow$ [(qd - 1)!·q] is divisible by p. But p and q are relatively prime integers, so (qd-1)! is divisible by p. Then $S(p) \le qd - 1$.

I prove that $S(p) \ge (q - 1)d$.

If we suppose against all reason that S(p) < (q - 1)d, it means [(q - 1)d - 1]! is divisible by p. Then (pd)| [(q - 1)d]! because d | (q - 1)d, so $S(pd) \le (q - 1)d$. This is contradiction with the fact that S(pd) = qd > (q - 1)d. We have the following inequalities:

$(q - 1)d \le S(p) \le qd - 1$.

For $q \ge 2$ we have from the first inequality $d \le \frac{S(p)}{q-1}$ and from the second $\frac{S(p+1)}{q} \le d$, so $\frac{S(p+1)}{q} \le d \le \frac{S(p)}{q-1}$.

For $k=\frac{q}{p}$, q ≥ 2, the equations has solutions if and only if there is a natural number between $\frac{S(p+1)}{q}$ and $\frac{S(p)}{q-1}$. If there isn't such a number, then the equation hasn't solutions. However, if there i a number d with $\frac{S(p+1)}{q} \le d \le \frac{S(p)}{q-1}$, this doesn't mean that the equation has solutions. This condition is necessary but not sufficient for the equation to have solutions.

For example:

a) $k=\frac{4}{5}$, q =4 , p=5 ⇒ $\frac{S(p+1)}{q}=\frac{6}{4}=\frac{3}{2}$, $\frac{S(p)}{q-1}=\frac{5}{3}$. In this case the equation hasn't solutions.

b) $k=\frac{3}{10}$, q=3, p=10 ; S(10)=5 , $\frac{6}{3}=2 \le d \le \frac{5}{2}$. If the equation has solutions, then we must have d=2 , n=dp=20 , S(n)=dq=6. But S(20)=5.

This is a contradiction. So there are no solutions for $h=\frac{3}{10}$.

We can ha·e more then natural numbers between $\frac{S(p+1)}{q}$ and $\frac{S(p)}{q-1}$. For example:

$k=\frac{3}{29}$, q=3 , p=29, $\frac{S(p+1)}{q}=10$, $\frac{S(p)}{q-1}=14{,}5$.

We prove that the equation $\frac{S(n)}{n}=k$ hasn't always solutions.

If q ≥ 2 then the number of solutions is equal with the number of values of d that verify relation (*). But d can be a nonnegative integer between $\frac{S(p+1)}{q}$ and $\frac{S(p)}{q-1}$, so d can take only a finite set of values. This means that the equation has no solutions or it has only a finite number of solutions.

We study note case $k=\frac{1}{p}$, p ∈ N*. In this case he equation has an infinite number of solutions. Let p_0 be a prime number such that $p<p_0$ and $n=pp_0$. We have $S(n)=S(pp_0)=p$, so $S(n)=p_0$. $\frac{S(n)}{n}=\frac{p_0}{pp_0}=\frac{1}{p}$, so the equation has an infinite number of solution.

I will refer now to another problem concerning the ratio $\frac{S(n)}{n}$ "Is there an infinity of natural numbers such that $0<\left\{\frac{x}{S(x)}\right\}<\left\{\frac{S(x)}{x}\right\}$?" from the same journal.

I will prove that the only number x that verifies the inequalities is x=9 : S(9)=6, $\frac{S(x)}{x}=\frac{6}{9}=\frac{2}{3}$, $\left\{\frac{x}{S(x)}\right\}=\left\{\frac{9}{6}\right\}=\frac{1}{2}$ and $0<\frac{1}{2}<\frac{2}{3}$, so x=9 verifies $0<\left\{\frac{x}{S(x)}\right\}<\left\{\frac{S(x)}{x}\right\}$.

Let $x=p_1^{\alpha_1}\dots p_n^{\alpha_n}$ be the standard form of x.

$S(x)=\max\limits_{1\le k\le n} S(p_k^{\alpha_k})$. We put $S(x)=S(p^{\alpha})$, where p^{α} is one of $p_1^{\alpha_1}\dots p_n^{\alpha_n}$ such that $S(p^{\alpha})=\max\limits_{1\le k\le n} S(p_k^{\alpha_k})$.

$\left\{\frac{x}{S(x)}\right\}$ can take one of the following values : $\frac{1}{S(x)}$, $\frac{2}{S(x)}$, ... , $\frac{S(x)-1}{S(x)}$ because $0 < \left\{\frac{x}{S(x)}\right\} < \left\{\frac{S(x)}{x}\right\}$ (We have $S(x) \le x$, so $\frac{S(x)}{x} \le 1$ and $\left\{\frac{S(x)}{x}\right\} \le \frac{S(x)}{x}$). This means $\frac{S(x)}{x} \ge \frac{1}{S(x)} \Rightarrow S(p^\alpha)^2 > x \ge p^\alpha$. (2)

But $(\alpha p)! = 1\cdot 2\cdot \ldots \cdot p(p+1)\ldots(2p)\ldots(\alpha p)$ is divisible by p^α , so $\alpha p \ge S(p^\alpha)$. From this last inequality and (2) it follows that $\alpha^2 p^2 > p^2$. We have three cases:

I. $\alpha=1$. In this case $S(x)=S(p)=p$, x is divisible by p, so $\frac{x}{p} \in Z$. This is a contradiction. There are no solutions for $\alpha=1$.

II. $\alpha=2$. In this case $S(x)=S(p^2)=2p$, because p is a prime number and $(2p)! = 1\cdot 2\cdot \ldots \cdot p(p-1)\ldots(2p)$, so $S(p^2)=2p$.

But $\left\{\frac{px_1}{2}\right\} \in \left\{0, \frac{1}{2}\right\}$. This means $\left\{\frac{px_1}{2}\right\} = \frac{1}{2} \Rightarrow \frac{1}{2} < \frac{2}{px_1} < 4$; p is a prime number $\Rightarrow p \in \{2,3\}$.

If p=2 and $px_1 < 4 \Rightarrow x_1 = 1$, but x=4 isn't a solution of the equation: S(4)=4 and $\left\{\frac{4}{4}\right\} = 0$.

If p=3 and $px_1 < 4 \Rightarrow x_1 = 1$, so $x=p^2=9$ is a solution of equation.

III. $\alpha=3$. We have $\alpha^2 p^2 > p^\alpha \Leftrightarrow \alpha^2 > p^{\alpha-1}$.

For $\alpha \ge 8$ we prove that we have $p^{\alpha-2} > p^2$, $(\forall)\ p \in N^*$, $p \ge 2$.

We prove by induction that $2^{n-1} > (n+1)^2$.

$2^{n-1} = 2\cdot 2^{n-2} \ge 2\cdot n^2 = n^2 + n^2 \ge n^2 + 8n > n^2 + 2n + 1 = (n+1)^2$, because $n \ge 8$.

We proved that $p^{\alpha-2} \ge 2^{\alpha-1} \ge \alpha^2$, for any $\alpha \ge 8$, $p \in N^*$, $p \ge 2$.

We have to study the case $\alpha \in \{ 3,4,5,6,7\}$.

a) $\alpha=3 \Rightarrow p \in \{ 2,3,5,7\}$, because p is a prime number.

If p=2 then $S(x)=S(2^3)=4$. But x is divisible by 8, so $\left\{\frac{x}{S(x)}\right\} = \left\{\frac{x}{4}\right\} = 0$, so x=4 cannot be a solution of the inequation.

If p=3 $\Rightarrow S(x)=S(3^3)=9$. But x is divisible by 27, so $\left\{\frac{x}{S(x)}\right\} = \left\{\frac{x}{9}\right\} = 0$, so x=9 cannot be a solution of the inequation.

If p=5 $\Rightarrow S(x)=S(5^3)=15$; $\left\{\frac{x}{S(x)}\right\} = \left\{\frac{S(x)}{x}\right\} = 0$ $x=5^3\cdot x_1$, $x_1 \in N^*$, $(5,x_1)=1$.

We have $0<\left\{\frac{5^2\cdot x_1}{3}\right\}<\left\{\frac{3}{5^2\cdot x_1}\right\}$. This first inequality implies $\left\{\frac{5^2\cdot x_1}{3}\right\}\in\left\{\frac{1}{3},\frac{2}{3}\right\}$, so $\frac{1}{3}<\frac{3}{5^2\cdot x_1}\Rightarrow 5^2\cdot x_1<9$, but this is impossible.

If $p=7\Rightarrow S(x)=S(7^3)=21$, $x=7^3\cdot x_1$, $(7,x_1)=1$, $x1\in N^*$.

We have $0<\left\{\frac{x}{S(x)}\right\}<\left\{\frac{S(x)}{x}\right\}\Rightarrow 0<\left\{\frac{7^2\cdot x_1}{3}\right\}<\frac{3}{7^2\cdot x_1}$. But $0<\left\{\frac{7^2\cdot x_1}{3}\right\}$ implies $\left\{\frac{7^2\cdot x_1}{3}\right\}\in\left\{\frac{1}{3},\frac{2}{3}\right\}$.

W have $\frac{1}{3}\leq\left\{\frac{7^2\cdot x_1}{3}\right\}\Rightarrow 7^2\cdot x_1<9$, but is impossible.

b) $\alpha=4:16\Rightarrow p\in\{2,3\}$.

If $p=2\Rightarrow S(x)=S(x^2)=6$, $x=16\cdot x_1$, $x_1\in N^*$, $(2,x_1)=1$, $0<\left\{\frac{x}{S(x)}\right\}<\frac{S(x)}{x}\Rightarrow 0<\left\{\frac{8x_1}{3}\right\}<\frac{3}{8x_1}$.

$0<\left\{\frac{8x_1}{3}\right\}\Rightarrow x_1=1\Rightarrow x=16$.

But $\frac{S(x)}{x}=\frac{6}{16}=\frac{3}{8}$; $\left\{\frac{x}{S(x)}\right\}=\left\{\frac{16}{6}\right\}=\left\{\frac{8}{3}\right\}=\frac{2}{3}\cdot\frac{2}{3}>\frac{3}{8}$, so the inequality isn't verified.

If $p=3\Rightarrow S(x)=S(3^4)=9$, $x=3^4\cdot x_1$, $(3,x_1)=1\Rightarrow 9|x\Rightarrow\frac{x}{S(x)}=0$, so the inequality isn't verified.

For $\alpha=\{5,6,7\}$, the only natural number $p>1$ that verifies the inequality $\alpha^2>p^{\alpha-2}$ is 2:

$\alpha=5:25>p^3\Rightarrow p=2$

$\alpha=6:36>p^4\Rightarrow p=2$

$\alpha=7:49>p$

In every case $x=2^\alpha\cdot x_1$, $x_1\in N^*$, $(x_1,2)=1$, and $S(x_1)\leq S(2^\alpha)$.

But $S(2^5)=S(2^6)=S(2^7)8$, so $S(x)=8$ But x is divisible by 8, so $\left\{\frac{x}{S(x)}\right\}=0$ so the inequality isn't verified because $0=\left\{\frac{x}{S(x)}\right\}$. We found that there is only x=9 to verify the inequality $0<\left\{\frac{x}{S(x)}\right\}<\left\{\frac{S(x)}{x}\right\}$

I try to study some diophantine equations proposed in "Smarandache Function Journal".

1) I study the equation $S(mx)=mS(x)$, $m\geq 2$ and x is a natural number.

Let x be a solution of the equation.

We have $S(x)!$ is divisible by x It is known that among m consecutive numbers, one is divisible by m, so $(S(x)!)$ is divisible by m, so $(S(x)+1)(S(x+2)...(S(x)+m)$ is divisible by (mx). We know that S(mx) is the smallest natural number such that $S(mx)!$ is divisible by (mx) and this implies $S(mx) \le S(x)+m$. But $S(mx)=mS(x)$, so $mS(x) \le S(x)+m \Leftrightarrow mS(x)-S(x)-m+1 \le \Leftrightarrow (m-1)(S(x)-1) \le 1$. We have several cases:

If m=1 then the equation becomes $S(X)=S(x)$, so any natural number is a solution of the equation.

If m=2, we have $S(x) \in \{1,2\}$ implies $x \in \{1,2\}$. We conclude that if m=1 then any natural number is a solution of the equation of the equation; if m=2 then x=1 and x=2 are only solution and if $m \ge 3$ the only solution of the equation is x=1.

2) Another equation is $S(x^y)=y^x$, x, y are natural numbers.

Let (x,y) be a solution of the equation.

$(yx)!=1...x(x+1)...(2x)...(yx)$ implies $S(x^y) \le yx$, so $y^x \le yx_1$ because $S(x^y)=y^x$.

But $y \ge 1$, so $y^{x-1} \le x$.

If x=1 then equation becomes $S(1) = y$, so y=1, so x=y=1 is a solution of the equation. If $x \ge 2$ then $x \ge 2^{x-1}$. But the only natural numbers that verify this inequality are x=y=2:

x=y=2 verifies the equation, so x=y=2 is a solution of the equation.

For $x \ge 3$ we prove that $x<2^{x-1}$. We make the proof by induction.

If x=3 : $3<2^{3-1}=4$.

We suppose that $k<2^{k-1}$ and we prove that $k+1<2^k$. We have $2^k=2 \cdot 2^k > 2 \cdot k = k+k > k+1$, so the inequality is established and there are no other solutions then x=y=1 and x=y=2.

3) I will prove that for any m,n natural numbers, if m>1 then the equation $S(x^n)=x^m$ has no solution or it has a finite number of solutions, and for m=1 the equation has a infinite number of solutions.

I prove that $S(x^n) \le nx$. But $x^m=S(x^n)$, so $x^m \le nx$.

For $m \ge 2$ we have $x^{m-1} \le n$. If m=2 then $x \le n$, and if $m \ge 3$ then $x \le \sqrt[m-1]{n}$, so x can take only a finite number of values, so the equation can have only a finite number of solutions or it has no solutions.

We notice that x=1 is a solution of the equation for any m,n natural numbers.

If the equation has a solution different of 1, we must have $x^m=S(x^n) \leq x^n$., so $m \leq n$

If m=n, the equation becomes $x^{m=n}=S(x^n)$, so x^n is a prime number or $x^n=4$, so n=1 and any prime number as well as x=4 is a solution of the equation, or n=2 and the only solutions are x=1 and x=2.

For m=1 and $n \geq 1$, we prove that the equations $S(x^m)=x$, $x \in N^*$ has an infinite number of solutions. Let be a prime number, p>n. We prove that)np) is a solution of the equation, that is $S((np)^n)=np$.

n<p and p is a prime number, so n and p are relatively prime numbers.

n<p implies:

$(np)! = 1 \cdot 2 \cdot \ldots \cdot n(n+1) \cdot \ldots \cdot (2n) \cdot \ldots \cdot (pn)$ is divisible by n^n.

$(np)! = 1 \cdot 2 \cdot \ldots \cdot p(p+1) \cdot \ldots \cdot (2p) \cdot \ldots \cdot (pn)$ is divisible by p^n.

But p and n are relatively prime numbers, so (np)! is divisible by $(np)^n$.

If we suppose that $S((np)^n)<np$, then we find that (np-1)! is a divisible by $(np)^n$, so(np-1)1 is divisible by p^n(3). But the exponent of p in the standard form of p in the standard form of (np-1)! is:

$$E = \left[\frac{np-1}{p}\right] + \left[\frac{np-1}{p^2}\right] + \ldots$$

But p >n, so $p^2 > np > np-1$. This implies :

$$\left[\frac{np-1}{p^k}\right] = 0$$, for any $k \geq 2$. We have:

$$E = \left[\frac{np-1}{p}\right] = n-1 .$$

This means (np-1)! is divisible by p^{n-1} , but isn't divisible by p^n , so this is a contradiction with (3). We proved that S((np)n)=np, so the equation S(xn)=x has an infinite number of solutions for any natural number n.

REFERENCE

Smarandache Function Journal, Vol.1 , 1990, Number Theory Publishing, Co, R. Muller Editor, Phoenix, New York, Lyon.

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