A result obtained using Smarandache Function

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Smarandache Function is defined as followed:
$S(m)$ = The smallest positive integer so that $S(m)!$ is divisible by m. [1]
Let's see the value which such function takes for $m = p^{p^n}$ with n integer, $n \geq 2$ and p prime number. To do so a Lemma is required.

Lemma 1 $\forall \quad m, n \in \mathbf{N} \quad m, n \geq 2$

$$m^n = E\left[\frac{m^{n+1} - m^n + m}{m}\right] + E\left[\frac{m^{n+1} - m^n + m}{m^2}\right] + \cdots + E\left[\frac{m^{n+1} - m^n + m}{m^{E[\log_m(m^{n+1} - m^n + m)]}}\right]$$

Where $E(x)$ gives the greatest integer less than or equal to x.

Demonstration:

Let's see in the first place the value taken by $E[\log_m(m^{n+1} - m^n + m)]$.

If $n \geq 2$: $m^{m+1} - m^n + m < m^{n+1}$ and therefore $\log_m(m^{n+1} - m^n + m) < \log_m m^{n+1} = n + 1$. As a result $E[\log_m(m^{n+1} - m^n + m)] < n + 1$.

And if $m \geq 2$: $mm^n \geq 2m^n \Rightarrow m^{n+1} \geq 2m^n \Rightarrow m^{n+1} + m \geq 2m^n \Rightarrow m^{n+1} - m^n + m \geq m^n \Rightarrow \log_m(m^{n+1} - m^n + m) \geq \log_m m^n = n \Rightarrow E[\log_m(m^{n+1} - m^n + m)] \geq n$

As a result: $n \leq E[\log_m(m^{n+1} - m^n + m)] < n + 1$ therefore:

$$E[\log_m(m^{n+1} - m^n + m)] = n \quad if\ n, m \geq 2$$

Now let's see the value which it takes for $1 \leq k \leq n$: $E\left[\frac{m^{n+1} - m^n + m}{m^k}\right]$

$$E\left[\frac{m^{n+1} - m^n + m}{m^k}\right] = E\left[m^{n+1-k} - m^{n-k} + \frac{1}{m^{k-1}}\right]$$

If $k = 1$: $E\left[\frac{m^{n+1} - m^n + m}{m}\right] = m^n - m^{n-1} + 1$

If $1 < k \leq n$: $E\left[\frac{m^{n+1} - m^n + m}{m^k}\right] = m^{n+1-k} - m^{n-k}$

Let's see what is the value of the sum:

$$\begin{array}{llllllll}
k=1 & m^n & -m^{n-1} & \cdots & \cdots & \cdots & \cdots & +1 \\
k=2 & & m^{n-1} & -m^{n-2} & & & & \\
k=3 & & & m^{n-2} & -m^{n-3} & & & \\
\vdots & & & & \vdots & & & \\
k=n-1 & & & & & m^2 & -m & \\
k=n & & & & & & m & -1
\end{array}$$

Therefore:

$$\sum_{k=1}^{n} E\left[\frac{m^{n+1}-m^n+m}{m^k}\right] = m^n \quad m,n \geq 2$$

Proposition 1 $\forall$ *p prime number* $\forall n \geq 2$:

$$S(p^{p^n}) = p^{n+1} - p^n + p$$

Demonstration:

Having $e_p(k)$ = exponent of the prime number p in the prime numbers descomposition of k.

We get:

$$e_p(k!) = E(\frac{k}{p}) + E(\frac{k}{p^2}) + E(\frac{k}{p^3}) + \cdots + E(\frac{k}{p^{E(\log_p k)}})$$

And using the Lemma we have:

$$e_p[(p^{n+1}-p^n+p)!] = E\left[\frac{p^{n+1}-p^n+p}{p}\right] + E\left[\frac{p^{n+1}-p^n+p}{p^2}\right] + \cdots + E\left[\frac{p^{n+1}-p^n+p}{m^{E[\log_p(p^{n+1}-p^n+p)]}}\right] = p^n$$

Therefore:

$$\frac{(p^{n+1}-p^n+p)!}{p^{p^n}} \in \mathbf{N} \quad and \quad \frac{(p^{n+1}-p^n+p-1)!}{p^{p^n}} \notin \mathbf{N}$$

And:

$$S(p^{p^n}) = p^{n+1} - p^n + p$$

References:

[1] C. Dumitrescu and R. Müller: *To Enjoy is a Permanent Component of Mathematics.* SMARANDACHE NOTIONS JOURNAL Vol. 9, No. 1-2.(1998) pp 21-26.

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