arXiv:2304.11759v1 [math.GR] 23 Apr 2023

RIGHT FEEBLE GROUPS

HIBA F. FAYOUMI AND HEE SIK KIM

Abstract. Right feeble groups are defined as groupoids $(X, *)$ such that (i) $x, y \in X$ implies the existence of $a, b \in X$ such that $a * x = y$ and $b * y = x$. Furthermore, (ii) if $x, y, z \in X$ then there is an element $w \in X$ such that $x * (y * z) = w * z$. These groupoids have a "remnant" group structure, which includes many other groupoids. In this paper, we investigate some properties of these groupoids. Enough examples are supplied to support the argument that they form a suitable class for systematic investigation.

1. Introduction

Among the many generalizations of the idea of groups already in the literature [2, 4, 7] we believe that the class of right feeble groups has not been discussed in its undisguised form until now. The terminology "feeble" comes from the fact that conditions (i) and (ii) defined in section 3 are replacements of the existence of inverses and the associative law. Indeed, suppose that the right feeble group $(X, *)$ contains an identity element e, where $x * e = e * x = x$ for all $x \in X$. Thus $a * x = e$ and $b * e = x$ implies $b = x$ and $a = x^{-1}$, i.e., the existence of inverses is assured. Also rule (ii) reads that for any x, y, z in the groupoid $(X, *)$ there is an element $w \in X$ such that $x * (y * z) = w * z$. For groups we find that $w = x * y$ and thus rule (ii) is a more general version of the associative law.

2. Preliminaries

Amid the various kinds of groupoids, leftoids play a special role. For instance, let $f : X \to X$ be any function, we define $(X, *, f)$ by the formula: $x * y := f(x)$. Thus, if we consider the groupoid-product $(X, *, f)\Box(X, \bullet, g) := (X, \Box)$, with $x\Box y = (x*y)\bullet(y*x) = f(x)\bullet g(y) =$

2010 Mathematics Subject Classification. 20N02.

Key words and phrases. right feeble group, groupoids, leftoids, rightoids, Smarandache disjoint, divisible, right entire, right asymmetric.

$g(f(x))$, it follows that $(X, \square) = (X, \square, g \circ f)$ is also a leftoid model of the composition of functions $(g \circ f)(x) = g(f(x))$. If $f(x) = x$, then $x * y = f(x) = x$ produces the left zero semigroup for which we obtain $(x * y) \bullet (y * x) = x \bullet y = x \square y$ if $(X, *)$ is the left zero semigroup. If $(X, \bullet)$ is the left zero semigroup, then $(x * y) \bullet (y * x) = x * y = x \square y$ as well. Meaning, the left zero semigroup acts like a multiplicative identity in $(Bin(X), \square)$ ([5]). H. Fayoumi ([3]) introduced the notion of the center $ZBin(X)$ in the semigroup $Bin(X)$ of all binary systems on a set X, and showed that if $(X, \bullet) \in ZBin(X)$, then $x \neq y$ implies $\{x, y\} = \{x \bullet y, y \bullet x\}$. Moreover, she showed that a groupoid $(X, \bullet) \in ZBin(X)$ if and only if it is a locally-zero groupoid.

Suppose that $f : X \to X$ is a surjection. If $x, y \in X$, then $f(a) = y$ and $f(b) = y$ for some choice of a and b, i.e., $a * b = f(a) = y$ and $b * y = f(b) = x$, so that condition (i) holds. Also $x * (y * z) = f(x)$ and $w * z = f(w)$. Since $x = w$ guarantees that $f(x) = f(w)$, it follows that condition (ii) holds as well and that the leftoid $(X, *, f)$ is a right feeble group.

Given a set X and a function $f : X \to X$, we consider a groupoid $(X, *, f)$, where $x * y := f(x)$ for any $x, y \in X$. Such a groupoid is called a *leftoid* over f ([5]). Similarly, we define a rightoid, i.e., $x * y := g(y)$ for all $x, y \in X$, where $g : X \to X$ is a function. Another idea of interest that will be useful in what follows is that of Smarandache disjointness. Given algebra types $(X, *)$ (type-P_1) and $(X, \circ)$ (type-P_2), we shall consider them to be *Smarandache disjoint* ([1]) if the following two conditions hold:

(A) If $(X, *)$ is a type-P_1-algebra with $|X| > 1$ then it cannot be a Smarandache-type-P_2-algebra $(X, \circ)$;
(B) If $(X, \circ)$ is a type-P_2-algebra with $|X| > 1$ then it cannot be a Smarandache-type-P_1-algebra $(X, *)$.

Thus, if K_1 and K_2 are two classes of mathematical objects, it may be that $K_1 \cap K_2$ consists precisely of one single object. Frequently this one single object is "trivial" in some way. For example, let K_1 be the class of d-algebras ([6]) and let K_2 be the class of groups. If $(X, *, 0)$ is both a d-algebra and a group with identity e, then $0 * x = 0$ implies $x = e$, the identity of the group, and thus $X = \{0\}$. Hence, $e = 0$ as well and $e * e = e = 0 = 0 * 0$.

Note that we may enlarge K_1 to the class of all groupoids $(X, *, 0)$ for which $0 * x = 0$ for all $x \in X$, to obtain the same conclusion. Similarly, K_2 may be enlarged to the class of groupoids $(X, *, e)$ where $a * x = a$ implies $x = e$. Hence $K_1 \cap K_2$ consists of the single groupoid $(X = \{u\}, u * u = u)$, and K_1 and K_2 are then Smarandache disjoint.

If $(X,*)$ is both a leftoid and a rightoid, then $x*y=C$, a constant from X. In this case, if $(X,*)$ has $x*y=C$ and $(X,\bullet)$ has $x\bullet y=D$, then the groupoids are isomorphic. Indeed, let $\varphi : X \to X$ be any bijection such that $\varphi(C)=D$, so that $\varphi(x*y)=\varphi(C)=D=\varphi(x)\bullet\varphi(y)$. Hence, leftoids and rightoids on a set X are Smarandache disjoint up to isomorphism of groupoids.

3. Right feeble groups

A groupoid $(X,*)$ is said to be a *right feeble group* if

(i) for any $x,y\in X$, there exist $a,b\in X$ such that $a*x=y$, and $b*y=x$,

(ii) for any $x,y,z\in X$, there exists $w\in X$ such that $x*(y*z)=w*z$.

Example 3.1. Let $\mathbb{R}$ be the set of all real numbers. If we define a binary operation "$*$" on $\mathbb{R}$ by $x*y:=\frac{1}{2}(x+y)$ for any $x,y\in\mathbb{R}$, then $(\mathbb{R},*)$ is a right feeble group. In fact, given $x,y\in\mathbb{R}$, if we take $a:=2y-x$, and $b:=2x-y$, then $a*x=y$ and $b*y=x$. Since $x*(y*z)=\frac{1}{2}(x+\frac{y+z}{2})$, we let $w:=x+\frac{1}{2}y-\frac{1}{2}z$, then $x*(y*z)=w*z$.

Note that $(\mathbb{R},*)$ in Example 3.1 is neither a group nor a semigroup. Assume that $(\mathbb{R},*)$ is a group with identity e. Then $x*e=x$ for all $x\in\mathbb{R}$. It follows that $\frac{x+e}{2}=x$ for all $x\in\mathbb{R}$, which shows that $x=e$ for all $x\in\mathbb{R}$, i.e., $|\mathbb{R}|=1$, a contradiction. Moreover, $1*(3*5)=\frac{5}{2}\neq\frac{7}{2}=(1*3)*5$.

Example 3.2. Let $(X,+,\cdot)$ be a field and let $\alpha,\beta,\gamma\in X$. If we define a binary operation "$*$" on X by $x*y:=\alpha+\beta x+\gamma y$ for any $x,y\in X$, then $(X,*)$ is a right feeble group.

In fact, for any $x,y\in X$, if we let $a:=\frac{1}{\beta}(y-\alpha-\gamma x)$, and $b:=\frac{1}{\beta}(x-\alpha-\gamma y)$, then it is easy to see that $a*x=y$ and $b*y=x$. Given $x,y,z,w\in X$, since $x*(y*z)=\alpha(1+\gamma)+\beta(x+y)+\gamma^2 z$ and $w*z=\alpha+\beta w+\gamma x$, if we take $w:=\frac{1}{\beta}[\alpha\gamma+\beta(x+y)+\gamma(\gamma-1)x]$, then $x*(y*z)=w*z$. This shows that $(X,*)$ is a right feeble group.

Proposition 3.3. *Every group is a right feeble group.*

Proof. Let $(X,*)$ be a group with identity e. Given $x,y\in X$, if we take $a:=y*x^{-1}$, and $b:=x*y^{-1}$, then $a*x=(y*x^{-1})*x=y*(x^{-1}*x)=y$ and $b*y=(x*y^{-1})*y=x*e=x$. Given $x,y,z\in X$, if we let $w:=x*y$, then $x*(y*z)=(x*y)*z=w*z$. Once again proving that $(X,*)$ is a right feeble group. $\square$

Proposition 3.4. Let $(X, *)$ be a leftoid for φ. If $\varphi(X) = X$, then $(X, *)$ is a right feeble group.

Proof. Given $x, y \in X$, since φ is onto, there exists $a \in X$ such that $\varphi(a) = y$. Since $(X, *)$ is a leftoid for φ, we have $a * x = \varphi(a) = y$. Similarly, if we take $b \in X$ such that $\varphi(b) = x$, then $b * y = \varphi(b) = x$.

Given $x, y, z \in X$, since $(X, *)$ is a leftoid for φ, we have $x * (y * z) = \varphi(x) = x * z$. □

The notion of Smarandache was introduced by Smarandache and Kandasamy in ([8]) studied the concept of Smarandache groupoids and Smarandache Bol groupoids. Padilla in ([9]) examined Smarandache algebraic structures. Allen, Kim and Neggers ([1]) introduced Smarandache disjointness in BCK/d-algebras. For more information on the notion of Smarandache we refer to ([8]).
In the next theorem, we consider the class of groups and the class of leftoids.

Theorem 3.5. *The class of groups and the class of leftoids are Smarandache disjoint.*

Proof. Let $(X, *)$ be both a leftoid for φ and a group. Then $e = x * x^{-1} = \varphi(x)$ for any $x \in X$, where e is the identity for the group $(X, *)$. It follows that $x * y = \varphi(x) = e = x * x$ for any $x, y \in X$. Since $(X, *)$ is a group, we obtain $x = y$ for all $x, y \in X$, proving that $|X| = 1$. □

Proposition 3.6. *Let $(X, *)$ be a right feeble group. If $f : (X, *) \to (Y, \bullet)$ is an epimorphism of groupoids, then $(Y, \bullet)$ is a right feeble group.*

Proof. Given $x, y \in Y$, since f is onto, there exist $a, b \in X$ such that $x = f(a)$, and $y = f(b)$. Since $(X, *)$ is a right feeble group, there exist $p, q, r, s \in X$ such that $p * a = b, q * b = a, r * b = a$, and $s * a = b$. It follows that $y = f(b) = f(p * a) = f(p) \bullet f(a) = f(p) \bullet x$ and $x = f(a) = f(q * b) = f(q) \bullet f(b) = f(q) \bullet y$.

Given $x, y, z \in Y$, since f is onto, there exist $a, b, c \in X$ such that $x = f(a), y = f(b)$, and $z = f(z)$. Since $(X, *)$ is a right feeble group, there exists $w \in X$ such that $a*(b*c) = w*c$. It follows that $x \bullet (y \bullet z) = f(a) \bullet (f(b) \bullet f(c)) = f(a * (b * c)) = f(w * c) = f(w) \bullet f(c) = f(w) \bullet z$, proving that $(Y, \bullet)$ is also a right feeble group. □

Proposition 3.7. *Let $(X, *)$, and $(Y, \bullet)$ be right feeble groups and let $Z := X \times Y$. Define $(x, y) \triangledown (u, v) := (x*u, y \bullet v)$ for all $(x, y), (u, v) \in Z$. Then $(Z, \triangledown)$ is also a right feeble group.*

Proof. Straightforward. □

Example 3.8. Let $\mathbb{R}$ be the set of all real numbers and "+" be the usual addition on X. Then $(\mathbb{R}, +)$ forms a group. By Proposition 3.2, it is a right feeble group. Let $A := [0, \infty)$. Then $(A, +)$ is a subgroupoid of $(\mathbb{R}, +)$, but not a right feeble group. In fact, if we assume $(A, +)$ is a right feeble group, then, for any $x, y \in A$, there exist $a, b \in \mathbb{R}$ such that $a + x = y$, and $b + y = x$. It follows that $y = a + x = a + (b + y)$. Since $a, b \in [0, \infty)$, we obtain $a = b = 0$, proving that $x = b + y = 0 + y = y$ for all $x, y \in A$, a contradiction. □

In Example 3.8, the subgroupoid $(A, +)$ is not a right feeble group. Hence, the following question: If $(A, *)$ is a subgroupoid of a right feeble group $(X, *)$, under what condition(s) will $(A, *)$ be a right feeble group?
To solve this problem, we introduce the notion of "divisibility".

Let $(X, *)$ be a groupoid. A subgroupoid $(A, *)$ is said to be *divisible* in $(X, *)$ if $a * x = y$ and $x, y \in A$, then $a \in A$.

Example 3.9. (a) Let $\mathbb{R}$ be the set of all real numbers and let $\mathbb{Q}$ be the set of all rational numbers without 0. Define a binary operation "$*$" on $\mathbb{R}$ by $x * y := xy$ (the usual multiplication). Assume that $a * x = y$ and $x, y \in \mathbb{Q}$. Since $x \neq 0$, we have $a = yx^{-1} \in \mathbb{Q}$, which shows that $(\mathbb{Q}, *)$ is divisible.

(b) Let $\mathbb{R}$ be the set of all real numbers and let $\mathbb{Z}$ be the set of all integers. Then $\frac{1}{2} \cdot 4 = 2$ and $4, 2 \in \mathbb{Z}$, but $\frac{1}{2} \notin \mathbb{Z}$, which shows that $(\mathbb{Z}, *)$ is not divisible.

If $(A, *)$ and $(B, *)$ are divisible in a groupoid $(X, *)$, and if $a * x = y$, where $x, y \in A \cap B$, then $a \in A \cap B$, and $A \cap B \neq \emptyset$ implies $(A \cap B, *)$ is divisible in $(X, *)$.

Theorem 3.10. *Let $(X, *)$ be a right feeble group. If $(A, *)$ is divisible in $(X, *)$, then $(A, *)$ is a right feeble group.*

Proof. (i) Given $x, y \in A$, since $(X, *)$ is a right feeble group, there exist $a, b \in X$ such that $a * x = y$, and $b * y = x$. Since $(A, *)$ is divisible and $x, y \in A$, we have $a, b \in A$ such that $a * x = y$, and $b * y = x$.
(ii) Given $x, y, z \in A$, we let $u := x * (y * z)$. Since $(A, *)$ is a subgroupoid of $(X, *)$, we have $u \in A$. Since A is divisible and $z \in A$, there exists $w \in A$ such that $w * z = u = x * (y * z)$. Hence, $(A, *)$ is a right feeble group. □

4. Right entire and right asymmetric

Let $(X, *)$ be a groupoid. We define a set $\rho(X, *)$ by

$$\rho(X, *) := \{x \in X | X * x = X\}.$$

Proposition 4.1. *If $(X, *)$ is a right feeble group, then $\rho(X, *) = X$.*

Proof. For any $x, y \in X$, since $(X, *)$ is a right feeble group, there exist $a, b \in X$ such that $y = a * x, x = b * y$. It follows that $y = a * x \in X * x$. Hence $X \subseteq X * x$, i.e., $X = X * x$, for all $x \in X$. This means that $x \in \rho(X, *)$ for all $x \in X$, proving that $X = \rho(X, *)$. □

The groupoid $(X, *)$ discussed in Proposition 4.1 is said to be *right entire*. It follows immediately that right feeble groups are right entire groups.

Proposition 4.2. *Let $(X, *), (Y, \bullet)$ be right entire groups and let $Z := X \times Y$. Define $(x, y) \triangledown (u, v) := (x * u, y \bullet v)$ for all $(x, y), (u, v) \in Z$. Then $(Z, \triangledown)$ is also a right entire groupoid.*

Proof. Straightforward. □

Proposition 4.3. *Let $(X, *)$ be a right entire groupoid. If $f : (X, *) \to (Y, \bullet)$ is an epimorphism of groupoids, then $(Y, \bullet)$ is a right entire groupoid.*

Proof. Given $x, y \in Y$, since f is onto, there exist $p, q \in X$ such that $x = f(q), y = f(p)$. Since $(X, *)$ is right entire, there exists $r \in X$ such that $r * p = q$ and hence $x = f(q) = f(r * p) = f(r) \bullet f(p) = f(r) \bullet y$. This shows that $Y = Y \bullet y$ for all $y \in Y$, proving that $\rho(Y, \bullet) = Y$. □

Proposition 4.4. *If $(X, *)$ is a leftoid for φ and right entire, then $\varphi(X) = X$.*

Proof. Let $(X, *)$ be a leftoid for φ and a right entire groupoid. Then $X * x = X$ for all $x \in X$, i.e., there exists $b \in X$ such that $a = b * x$ for any $a \in X$. Since $(X, *)$ is a leftoid for φ, we have $a = b * x = \varphi(b) \in \varphi(X)$ for all $a \in X$, proving that $X \subseteq \varphi(X)$. □

A groupoid $(X, *)$ is said to be *right asymmetric* if $a * x = y$, and $b * y = x$ for some $a, b \in X$, then $x = y$.

Example 4.5. Let $X := [0, \infty)$ and let $x * y := x + y$ for all $x, y \in X$. Then $(X, *)$ is right asymmetric. In fact, if $a * x = y$, and $b * y = x$, then $a + x = y$, and $b + y = x$, and hence $y = a + x = a + (b + y) = (a + b) + y$. It follows that $a + b = 0$, i.e., $a = b = 0$. This shows that $x = y$.

Theorem 4.6. *The class of right entire groupoids and the class of right asymmetric groupoids are Smarandache disjoint.*

Proof. Assume $(X, *)$ be both a right entire groupoid and a right asymmetric groupoid. Then $X * x = X, X * y = X$ for all $x, y \in X$. It follows that $a * x = y$, and $b * y = x$ for some $a, b \in X$. Since $(X, *)$ is right asymmetric, we obtain $x = y$ for all $x, y \in X$. Hence, $|X| = 1$. □

5. Some relations

Given a groupoid $(X, *)$, we define a binary operation "$\leq$" on X by

$$x \leq y \iff \exists\, a \in X \text{ s.t. } a * x = y.$$

Proposition 5.1. *If $(X, *)$ is a right entire groupoid, then $\leq$ is reflexive.*

Proof. Assume $(X, *)$ is right entire. Then $\rho(X, *) = X$, i.e., $X * x = X$ for all $x \in X$. It follows that there exists $a \in X$ such that $x = a * x$ for any $x \in X$, i.e., $x \leq x$ for any $x \in X$. □

Proposition 5.2. *A groupoid $(X, *)$ is right asymmetric if and only if $\leq$ is anti-symmetric.*

Proof. Let $(X, *)$ be a right asymmetric groupoid. Assume $x \leq y$ and $y \leq x$. Then there exist $a, b \in X$ such that $a * x = y$, and $b * y = x$. Since $(X, *)$ is right asymmetric, we obtain $x = y$. The converse is trivial and we omit the proof. □

Proposition 5.3. *If $(X, *)$ is a right feeble group, then $\leq$ is transitive.*

Proof. Assume that $x \leq y$, and $y \leq z$. Then there exist $a, b \in X$ such that $a * x = y$, and $b * y = z$. Since $(X, *)$ is a right feeble group, there exists $c \in X$ such that $z = b * y = b * (a * x) = c * x$, which shows that $x \leq z$. □

Proposition 5.4. *Let $(X, *)$ be a groupoid with $e \in X$ such that $e * x = x$ for all $x \in X$. Then $x \leq x$ for all $x \in X$.*

Note that '$(X, *)$ has a left identity' does not mean $X * x = X$ for some $x \in X$. For example, let $N := \{0, 1, 2, \cdots\}$. Then $(N, +)$ has an identity 0 and $0 + x = x$ for all $x \in N$, but $N + 2 = \{2, 3, \cdots\} \neq N$.

Proposition 5.5. *Let $(X, *)$ be a groupoid and let $a \in X$ such that $a * X = X$. Then there exists $x \in X$ such that $x \leq y$ for any $y \in X$.*

Proof. If $a * X = X$, then there exists $x \in X$ such that $y = a * x$ for any $y \in X$. It follows that $x \leq y$. □

Let $N := \{0, 1, 2, \cdots\}$. Then $0 + N = N$ and hence $y \leq y$ for all $y \in N$.

Proposition 5.6. *Let $(A, *)$ be a divisible subgroupoid of a groupoid $(X, *)$. If $x, y \in A$ such that $x \leq y$ in $(X, *)$, then $x \leq y$ in $(A, *)$.*

Proof. Let $x, y \in A$ such that $x \leq y$ in $(X, *)$. Then there exists $a \in X$ such that $a * x = y$. Since $(A, *)$ is divisible and $x, y \in A$, we have $a \in A$, i.e., $x \leq y$ in $(A, *)$. □

Theorem 5.7. *Let $(X, *)$ be a right entire groupoid. If $(A, *)$ is a divisible subgroupoid of $(X, *)$, then $(A, *)$ is right entire.*

Proof. Given $a \in A$, since $(X, *)$ is right entire, we have $X * a = X$. It follows that there exists $y \in X$ such that $x = y * a$ for any $x \in A$. Since $(A, *)$ is divisible, we obtain $y \in A$ and hence $x = y * a \in A * a$. Hence $A \subseteq A * a$ for all $a \in A$. Clearly, $A * a \subseteq A$, proving that $A = A * a$ for any $a \in A$. □

Proposition 5.8. *Let $(X, *)$ be a right asymmetric groupoid. If $(A, *)$ is a divisible groupoid in $(X, *)$, then $(A, *)$ is also right asymmetric.*

Proof. If $x, y \in A$, then $x, y \in X$. Since $(X, *)$ is right asymmetric, if $a * x = y$, and $b * y = x$ for some $a, b \in X$, then $x = y$. We show that $a, b \in A$. Consider $a * x = y$. Since $x, y \in A$ and A is divisible, we obtain $a \in A$. Similarly, $b \in A$. Hence $(A, *)$ is right asymmetric. □

Proposition 5.9. *Let $(X, *)$ be a group with identity e. Then every subgroup $(A, *)$ of $(X, *)$ is divisible.*

Proof. Let $x, y \in A$ such that $a * x = y$ for some $a \in X$. It follows that $a = y * x^{-1} \in A$ since $(A, *)$ is a subgroup of X. This shows that $(A, *)$ is divisible in $(X, *)$. □

References

[1] P. J. Allen, Hee Sik Kim and J. Neggers, *Smarandache disjoint in BCK/d-algebras*, Sci. Math. Japo. **61** (2005), 447-449.

[2] P. J. Allen, Hee Sik Kim and J. Neggers, *Several types of groupoids induced by two-variable functions*, Springer Plus **5** (2016), 1715-1725.

[3] H. Fayoumi, *Locally-zero groupoids and the center of $Bin(X)$*, Comm. Korean Math. Soc. **26** (2011), 163-168.

[4] M. H. Hooshmand, *Grouplikes*, Bull. Iranian Math. Soc. **39** (2013), 65-86.

[5] Hee Sik Kim and J. Neggers, *The semigroups of binary systems and some perspectives*, Bull. Korean Math. Soc. **45** (2008), 651-661.

[6] J. Neggers and Hee Sik Kim, *On d-algebras*, Math. Slovaca **49** (1999), 19-26.

[7] G. Wu and Y. H. Kim, *Pregroups and Pre-B-algebras*, J. Appl. Math. & Inf. **36** (2018), 51-57.

[8] W. B. V. Kandasamy, *Groupoids and Smarandache Groupoids*, American Research Press, Rehoboth, 2002.

[9] R. Padilla, *Smarandache algebraic structures*, Bull. Pure & Appl. Sci. **17** (1988), 119-121.

H. Fayoumi, Department of Mathematics and Statistics, University of Toledo, Toledo, OH 43606-3390, U. S. A.

E-mail address: hiba.fayoumi@UToledo.edu

Hee Sik Kim, Department of Mathematics, Hanyang University, Seoul, 04763, Korea

E-mail address: heekim@hanyang.ac.kr