Smarandache Curves and Spherical Indicatrices in the Galilean 3-Space

H.S.Abdel-Aziz and *M.Khalifa Saad**

Dept. of Math., Faculty of Science, Sohag Univ., 82524 Sohag, Egypt

Abstract. In the present paper, Smarandache curves for some special curves in the three-dimensional Galilean space $G^3$are investigated. Moreover, spherical indicatrices for the helix as well as circular helix are introduced. Furthermore, some properties for these curves are given. Finally, in the light of this study, some related examples of these curves are provided.

Keywords. Smarandache curves, Spherical indicatrices, Frenet frame, Galilean space.

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1 Introduction

Discovering Galilean space-time is probably one of the major achievements of non relativistic physics. Nowadays Galilean space is becoming increasingly popular as evidenced from the connection of the fundamental concepts such as velocity, momentum, kinetic energy, etc. and principles as indicated in [7]. In recent years, researchers have begun to investigate curves and surfaces in the Galilean space and thereafter pseudo-Galilean space. A regular curve in Euclidean space whose position vector is composed by Frenet frame vectors on another regular curve is called a Smarandache curve. Smarandache curves have been investigated by some differential geometers [1, 9]. M. Turgut and S. Yilmaz have defined a special case of such curves and call it Smarandache TB_2 curves in the space E_1^4 [9]. They have dealed with a special Smarandache curves which is defined by the tangent and second binormal vector fields. Additionally, they have computed formulas of this kind curves. In [1], the author has introduced some special Smarandache curves in the Euclidean space. He has studied Frenet-Serret invariants of a special case. In this paper, we investigate Smarandache curves for the position vector of an arbitrary curve in terms of the curvature and the torsion with respect to standard frame. Besides, special Smarandache curves such as TN, TB and TNB according to Frenet frame in Galilean 3-space are studied. Furthermore, we find differential geometric properties of these special curves and we calculate curvatures (natural curvatures) of these curves. Our main result in this work is to study Smarandache curves for some special curves in the Galilean 3-space G^3.

* E-mail address: mohamed_khalifa77@science.sohag.edu.eg

2 Preliminaries

Let us recall the basic facts about the three-dimensional Galilean geometry G^3. The geometry of the Galilean space has been firstly explained in [11]. The curves and some special surfaces in G^3 are considered in [3]. The Galilean geometry is a real Cayley-Klein geometry with projective signature $(0,0,+,+)$ according to [5]. The absolute of the Galilean geometry is an ordered triple (w,f,I) where w is the ideal (absolute) plane $(x_0 = 0)$, f is a line in w $(x_0 = x_1 = 0)$ and I is elliptic $((0:0:x_2:x_3) \longrightarrow (0:0:x_3:-x_2))$ involution of the points of f. In the Galilean space there are just two types of vectors, non-isotropic $\mathbf{x}(x,y,z)$ (for which holds $x \neq 0$). Otherwise, it is called isotropic. We do not distinguish classes of vectors among isotropic vectors in G^3. A plane of the form $x = const.$ in the Galilean space is called Euclidean, since its induced geometry is Euclidean. Otherwise it is called isotropic plane. In affine coordinates, the Galilean inner product between two vectors $P = (p_1,p_2,p_3)$ and $Q = (q_1,q_2,q_3)$ is defined by [4]:

$$\langle P,Q\rangle_{G^3} = \begin{cases} p_1q_1 & \text{if } p_1 \neq 0 \vee q_1 \neq 0, \\ p_2q_2 + p_3q_3 & \text{if } p_1 = 0 \wedge q_1 = 0. \end{cases} \tag{2.1}$$

And the cross product in the sense of Galilean space is given by:

$$(P\times Q)_{G^3} = \begin{cases} \begin{vmatrix} 0 & e_2 & e_3 \\ p_1 & p_2 & p_3 \\ q_1 & q_2 & q_3 \end{vmatrix}; & \text{if } p_1 \neq 0 \vee q_1 \neq 0, \\ \\ \begin{vmatrix} e_1 & e_2 & e_3 \\ p_1 & p_2 & p_3 \\ q_1 & q_2 & q_3 \end{vmatrix}; & \text{if } p_1 = 0 \wedge q_1 = 0. \end{cases} \tag{2.2}$$

The Galilean sphere of radius d and center c of the space G^3 is defined by

$$S^2_G(c,d) = \{X - c \in G^3 : \langle X-c, X-c\rangle_{G^3} = \pm d^2\}.$$

A curve $\alpha(t) = (x(t),y(t),z(t))$ is admissible in G^3 if it has no inflection points $(\dot{\alpha}(t)\times\ddot{\alpha}(t) \neq 0)$. An admissible curve in G^3 is an analogue of a regular curve in Euclidean space.

For an admissible curve $\alpha : I \rightarrow G^3$, $I \subset R$ parameterized by the arc length s with differential form $dt = ds$, given by

$$\alpha(s) = (s, y(s), z(s)). \tag{2.3}$$

The curvature $\kappa(s)$ and torsion $\tau(s)$ of α are defined by

$$\begin{aligned} \kappa(s) &= \left\|\alpha''(s)\right\| = \sqrt{y''(s)^2 + z''(s)^2}, \\ \tau(s) &= \frac{y''(s)z'''(s) + y'''(s)z''(s)}{\kappa^2(s)}. \end{aligned} \tag{2.4}$$

Note that an admissible curve has non-zero curvature.

The associated trihedron is given by

$$\begin{aligned}\mathbf{T}(s) &= \alpha^{'}(s)=(1,y^{'}(s),z^{'}(s)),\\ \mathbf{N}(s) &= \frac{\alpha^{''}(s)}{\kappa(s)}=\frac{(0,y^{''}(s),z^{''}(s))}{\kappa(s)},\\ \mathbf{B}(s) &= \frac{(0,-z^{''}(s),y^{''}(s))}{\kappa(s)}.\end{aligned} \tag{2.5}$$

For derivatives of the tangent $\mathbf{T}$, normal $\mathbf{N}$ and binormal $\mathbf{B}$ vector field, the following Frenet formulas in the Galilean space hold [11]

$$\begin{bmatrix}\mathbf{T}\\ \mathbf{N}\\ \mathbf{B}\end{bmatrix}^{'}=\begin{bmatrix}0&\kappa&0\\ 0&0&\tau\\ 0&-\tau&0\end{bmatrix}\begin{bmatrix}\mathbf{T}\\ \mathbf{N}\\ \mathbf{B}\end{bmatrix}. \tag{2.6}$$

From (2.5) and (2.6), we derive an important relation

$$\alpha'''(s)=\kappa'(s)\mathbf{N}(s)+\kappa(s)\tau(s)\mathbf{B}(s).$$

Let us consider the following definitions:

Definition 2.1 *As in the three-dimensional Euclidean space, an admissible curve in G^3, whose position vector is composed by Frenet frame vectors on another admissible curve is called a Smarandache curve [9].*

In the light of the above definition, we adapt it to admissible curves in the Galilean space as follows:

Definition 2.2 *let $\Gamma=\Gamma(s)$ be an admissible curve in G^3 and $\{\mathbf{T},\mathbf{N},\mathbf{B}\}$ be its moving Frenet frame. Smarandache $\mathbf{TN},\mathbf{TB}$ and $\mathbf{TNB}$ curves are respectively, defined by*

$$\begin{aligned}\Gamma_{\mathbf{TN}} &= \frac{\mathbf{T}+\mathbf{N}}{\|\mathbf{T}+\mathbf{N}\|},\\ \Gamma_{\mathbf{TB}} &= \frac{\mathbf{T}+\mathbf{B}}{\|\mathbf{T}+\mathbf{B}\|},\\ \Gamma_{\mathbf{TNB}} &= \frac{\mathbf{T}+\mathbf{N}+\mathbf{B}}{\|\mathbf{T}+\mathbf{N}+\mathbf{B}\|}.\end{aligned} \tag{2.7}$$

Definition 2.3 *A helix is a geometric curve with non-vanishing constant curvature κ and non-vanishing constant torsion τ [10]*

Definition 2.4 *A family of curves with constant curvature but non-constant torsion is called Salkowski curves and a family of curves with constant torsion but non-constant curvature is called Anti-Salkowski curves [6].*

Definition 2.5 *As in Euclidean 3-space , let α be an admissible curve in Galilean 3-space with Frenet vectors $\mathbf{T},\mathbf{N}$ and $\mathbf{B}$. The unit tangent vectors along the curve α generate a curve $\alpha_{\mathbf{T}}$ on a Galilean* *sphere of radius 1 about the origin. The curve $\alpha_{\mathbf{T}}$ is called the spherical indicatrix of* **T** *or more commonly, $\alpha_{\mathbf{T}}$ is called tangent indicatrix of the curve α. If $\alpha = \alpha(s)$ is a natural representation of α, then $\alpha_{\mathbf{T}}(s_{\mathbf{T}}) = \mathbf{T}(s)$ will be a representation of $\alpha_{\mathbf{T}}$. Similarly one considers the principal normal indicatrix $\alpha_{\mathbf{N}}(s_{\mathbf{N}}) = \mathbf{N}(s)$ and binormal indicatrix $\alpha_{\mathbf{B}}(s_{\mathbf{B}}) = \mathbf{B}(s)$ [8].*

3 Smarandache curves of special curves

In this section, we consider the position vector of an arbitrary curve with curvature $\kappa(s)$ and torsion $\tau(s)$ in the Galilean space G^3 which computed from the natural representation form as follows [2]

$$r(s) = \left[s, \int\left[\int \kappa(s)\cos\left[\int \tau(s)ds\right]ds\right]ds, \int\left[\int \kappa(s)\sin\left[\int \tau(s)ds\right]ds\right]ds\right]. \tag{3.1}$$

Its moving frame is

$$\begin{aligned} \mathbf{T}(s) &= \left[1, \left[\int \kappa(s)\cos\left[\int \tau(s)ds\right]ds\right], \left[\int \kappa(s)\sin\left[\int \tau(s)ds\right]ds\right]\right], \\ \mathbf{N}(s) &= \left[0, \cos\left[\int \tau(s)ds\right], \sin\left[\int \tau(s)ds\right]\right], \\ \mathbf{B}(s) &= \left(0, -\sin\left[\int \tau(s)ds\right], \cos\left[\int \tau(s)ds\right]\right). \end{aligned} \tag{3.2}$$

Let us investigate Frenet invariants of Smarandache curves according to definition 2.2.

3.1 TN-Smarandache curve

Let $r_1(s_1)$ be a **TN**-Smarandache curve of $r(s)$written as

$$r_1(s_1) = \left(\begin{array}{c} 1, \int \kappa(s)\cos\left[\int \tau(s)ds\right]ds + \cos\left[\int \tau(s)ds\right], \\ \int \kappa(s)\sin\left[\int \tau(s)ds\right]ds + \sin\left[\int \tau(s)ds\right] \end{array}\right). \tag{3.3}$$

By differentiating (3.3) with respect to s, we get

$$\mathbf{T}_1 = \frac{1}{\sqrt{\kappa^2+\tau^2}}\left[\begin{array}{c} 0, \kappa(s)\cos\left[\int \tau(s)ds\right] - \tau(s)\sin\left[\int \tau(s)ds\right], \\ \kappa(s)\sin\left[\int \tau(s)ds\right] + \tau(s)\cos\left[\int \tau(s)ds\right] \end{array}\right], \tag{3.4}$$

where

$$\frac{ds_1}{ds} = \sqrt{\kappa^2+\tau^2}. \tag{3.5}$$

Also, differentiating (3.4), we obtain

$$\kappa_1\mathbf{N}_1 = \frac{1}{(\kappa^2+\tau^2)^2}\left(0, \lambda_1, \lambda_2\right), \tag{3.6}$$

then, the curvature and principal normal vector field of $r_1(s_1)$ are respectively,

$$\kappa_1 = \frac{1}{(\kappa^2+\tau^2)^2}\sqrt{\lambda_1^2+\lambda_2^2}, \quad \mathbf{N}_1 = \frac{1}{\sqrt{\lambda_1^2+\lambda_2^2}}\left(0, \lambda_1, \lambda_2\right), \tag{3.7}$$

where

$$\lambda_1 = \left(-\kappa^3\tau - \kappa^2\dot{\tau} - \tau^3\kappa + \kappa\dot{\kappa}\tau\right)\sin\left[\int \tau(s)ds\right] + \left(-\kappa^2\tau^2 + \tau^2\dot{\kappa} - \tau^4 - \dot{\tau}\kappa\tau\right)\cos\left[\int \tau(s)ds\right], \tag{3.8}$$

$$\lambda_2 = \left(\kappa^3\tau + \kappa^2\dot{\tau} + \tau^3\kappa - \kappa\dot{\kappa}\tau\right)\cos\left[\int \tau(s)ds\right] + \left(\kappa^2\tau^2 - \tau^2\dot{\kappa} + \tau^4 + \dot{\tau}\kappa\tau\right)\sin\left[\int \tau(s)ds\right]. \tag{3.9}$$

On the other hand, the binormal vector of this curve is expressed as

$$\mathbf{B}_1 = \frac{1}{\sqrt{(\kappa^2+\tau^2)\left(\lambda_1^2+\lambda_2^2\right)}} \left[\begin{array}{c} \lambda_2\left(\kappa(s)\cos\left[\int\tau(s)ds\right] - \tau(s)\sin\left[\int\tau(s)ds\right]\right) - \\ \lambda_1\left(\kappa(s)\sin\left[\int\tau(s)ds\right] + \tau(s)\cos\left[\int\tau(s)ds\right]\right), 0, 0 \end{array}\right], \tag{3.10}$$

it follows that

$$\tau_1 = 0. \tag{3.11}$$

3.2 TB-Smarandache curve

Let $r_2(s_2)$ be a **TB**-Smarandache curve given by

$$r_2(s_2) = \left[\begin{array}{c} 1, \int\kappa(s)\cos\left[\int\tau(s)ds\right] - \sin\left[\int\tau(s)ds\right], \\ \int\kappa(s)\sin\left[\int\tau(s)ds\right] + \cos\left[\int\tau(s)ds\right] \end{array}\right]. \tag{3.12}$$

Differentiating this equation with respect to s yields

$$\mathbf{T}_2 = \left(0, \cos\left[\int \tau(s)ds\right], \sin\left[\int \tau(s)ds\right]\right), \tag{3.13}$$

where

$$\frac{ds_2}{ds} = (\kappa - \tau). \tag{3.14}$$

Again, differentiating (3.13) gives

$$\kappa_2 = \frac{\tau}{(\kappa-\tau)}, \tag{3.15}$$

$$\mathbf{N}_2 = \left(0, -\sin\left[\int \tau(s)ds\right], \cos\left[\int \tau(s)ds\right]\right). \tag{3.16}$$

From the above mentioned, one can obtain

$$\mathbf{B}_2 = (1,0,0). \tag{3.17}$$

From which

$$\tau_2 = 0. \tag{3.18}$$

3.3 TNB-Smarandache curve

By definition 2.2, the **TNB**-Smarandache curve is given by

$$r_3(s_3) = \begin{bmatrix} 1, \int \kappa(s)\cos\left[\int \tau(s)ds\right] + \cos\left[\int \tau(s)ds\right] - \sin\left[\int \tau(s)ds\right], \\ \int \kappa(s)\sin\left[\int \tau(s)ds\right] + \sin\left[\int \tau(s)ds\right] + \cos\left[\int \tau(s)ds\right] \end{bmatrix}. \tag{3.19}$$

Similarly, its tangent, the principal normal and the binormal vectors are respectively,

$$\mathbf{T}_3 = \frac{1}{\sqrt{(\kappa-\tau)^2+\tau^2}} \begin{bmatrix} 0, (\kappa-\tau)\cos\left[\int \tau(s)ds\right] - \tau(s)\sin\left[\int \tau(s)ds\right], \\ (\kappa-\tau)\sin\left[\int \tau(s)ds\right] + \tau(s)\cos\left[\int \tau(s)ds\right] \end{bmatrix}, \tag{3.20}$$

in which

$$\frac{ds_3}{ds} = \sqrt{(\kappa-\tau)^2+\tau^2}, \tag{3.21}$$

$$\mathbf{N}_3 = \frac{\begin{pmatrix} 0, \mu_1\cos\left[\int \tau(s)ds\right] - \mu_2\sin\left[\int \tau(s)ds\right], \\ \mu_2\cos\left[\int \tau(s)ds\right] + \mu_1\sin\left[\int \tau(s)ds\right] \end{pmatrix}}{\sqrt{\mu_1^2+\mu_2^2}}, \tag{3.22}$$

$$\mathbf{B}_3 = \frac{(\mu_2(\kappa-\tau) - \tau\mu_1)}{\sqrt{(\kappa-\tau)^2+\tau^2}\sqrt{\mu_1^2+\mu_2^2}}(1,0,0), \tag{3.23}$$

where

$$\begin{aligned} \mu_1(s) &= \left(\left((\kappa-\tau)^2+\tau^2\right)\left(\dot\kappa-\dot\tau-\tau^2\right) - (\kappa-\tau)\left((\kappa-\tau)(\dot\kappa-\dot\tau)+\tau\dot\tau\right)\right), \\ \mu_2(s) &= \left(\left((\kappa-\tau)^2+\tau^2\right)\left(\tau\kappa-\tau^2+\dot\tau\right) - \tau\left((\kappa-\tau)(\dot\kappa-\dot\tau)+\tau\dot\tau\right)\right). \end{aligned} \tag{3.24}$$

The curvature and the torsion of this curve are respectively, given by

$$\kappa_3 = \frac{\sqrt{\mu_1^2+\mu_2^2}}{\left((\kappa-\tau)^2+\tau^2\right)^2}, \quad \tau_3 = 0. \tag{3.25}$$

Theorem 3.1 *Let $r(s)$ given by (3.1) be a helix $\beta(s)$ in G^3 ($\tau/\kappa = m = const.$)which can be written as*

$$\beta(s) = \left(s, \frac{1}{m}\int \sin\left[m\int \kappa(s)ds\right]ds, -\frac{1}{m}\int \cos\left[m\int \kappa(s)ds\right]ds\right), \tag{3.26}$$

then **TN**, **TB** *and* **TNB**-*Smarandache curves of β are plane curves with* constant *curvatures.*

Proof. We prove this theorem for **TN**-Smarandache curve only. The straightforward computations on β give respectively, the tangent, the principal normal and the binormal vectors of β as follows

$$\mathbf{T}_\beta(s) = \left(1, \frac{1}{m}\sin\left[m\int \kappa(s)ds\right], -\frac{1}{m}\cos\left[m\int \kappa(s)ds\right]\right),$$

$$\mathbf{N}_\beta(s) = \left(0, \cos\left[m\int \kappa(s)ds\right], \sin\left[m\int \kappa(s)ds\right]\right),$$

$$\mathbf{B}_\beta(s) = \left(0, -\sin\left[m\int\kappa(s)ds\right], \cos\left[m\int\kappa(s)ds\right]\right). \tag{3.27}$$

The **TN**-Smarandache curve of β can be expressed as

$$\beta_1(s_1) = \left(\begin{array}{c} 1, \frac{1}{m}\sin\left[m\int\kappa(s)ds\right] + \cos\left[m\int\kappa(s)ds\right], \\ -\frac{1}{m}\cos\left[m\int\kappa(s)ds\right] + \sin\left[m\int\kappa(s)ds\right] \end{array}\right). \tag{3.28}$$

It has the moving Frenet frame

$$\mathbf{T}_{\beta_1} = \frac{1}{\sqrt{1+m^2}}\left(\begin{array}{c} 0, \cos\left[m\int\kappa(s)ds\right] - m\sin\left[m\int\kappa(s)ds\right], \\ \sin\left[m\int\kappa(s)ds\right] + m\ \cos\left[m\int\kappa(s)ds\right] \end{array}\right),$$

$$\mathbf{N}_{\beta_1} = \frac{1}{(1+m^2)}\left(\begin{array}{c} 0, -\sin\left[m\int\kappa(s)ds\right] - m\cos\left[m\int\kappa(s)ds\right], \\ \cos\left[m\int\kappa(s)ds\right] - m\sin\left[m\int\kappa(s)ds\right] \end{array}\right),$$

$$\mathbf{B}_{\beta_1} = \left(\frac{1}{\sqrt{1+m^2}}, 0, 0\right). \tag{3.29}$$

From the above data, the curvature and torsion of β_1 are

$$\kappa_{\beta_1} = m,\ \tau_{\beta_1} = 0. \tag{3.30}$$

The curvature is constant and torion is vanished. The same results for **TB** and **TNB**-Smarandache curves of β are valid. Hence the proof is completed. ■

Remark 3.1 *If $r(s)$ is a circular helix ($\kappa = const.$, $\tau = const.$), then* **TN**, **TB** *and* **TNB**-*Smarandache curves are also plane curves with constant curvatures.*

Theorem 3.2 *If $r(s)$ is a family of Salkowski curves $\gamma(s)$ in G^3 ($\tau = \tau(s), \kappa = a = const.$)which can be written as*

$$\gamma(s) = \left(s, a\int\left[\int\cos\left(\int\tau(s)ds\right)ds\right]ds, a\int\left[\int\sin\left(\int\tau(s)ds\right)ds\right]ds\right), \tag{3.31}$$

then **TN**, **TB** *and* **TNB**-*Smarandache curves of γ are plane curves with* non-constant *curvatures.*

Proof. After some calculations on γ, the tangent, the principal normal and the binormal vectors of γ are, respectively

$$\mathbf{T}_\gamma(s) = \left(1, a\int\left[\cos\left(\int\tau(s)ds\right)\right]ds, a\int\left[\sin\left(\int\tau(s)ds\right)\right]ds\right),$$

$$\mathbf{N}_\gamma(s) = \left(0, \cos\left[\int\tau(s)ds\right], \sin\left[\int\tau(s)ds\right]\right),$$

$$\mathbf{B}_\gamma(s) = \left(0, -\sin\left[\int\tau(s)ds\right], \cos\left[\int\tau(s)ds\right]\right). \tag{3.32}$$

Here, **TN**-Smarandache curve of γ is

$$\gamma_1(s_1) = \left(\begin{array}{c} 1, a\int\left[\cos\left(\int\tau(s)ds\right)\right]ds + \cos\left[\int\tau(s)ds\right], \\ a\int\left[\sin\left(\int\tau(s)ds\right)\right]ds + \sin\left[\int\tau(s)ds\right] \end{array}\right), \tag{3.33}$$

with the Frenet vectors

$$\mathbf{T}_{\gamma_1} = \frac{1}{\sqrt{a^2-\tau^2(s)}}\left(\begin{array}{c} 0, a\cos(\int\tau(s)ds) - \tau(s)\sin(\int\tau(s)ds), \\ a\sin(\int\tau(s)ds) + \tau(s)\cos(\int\tau(s)ds) \end{array}\right),$$

$$\mathbf{N}_{\gamma_1} = \frac{1}{\Omega}\left(\begin{array}{c} 0, -\theta_1(s)\sin(\int\tau(s)ds) + \theta_2(s)\cos(\int\tau(s)ds), \\ \theta_2(s)\sin(\int\tau(s)ds) + \theta_1(s)\cos(\int\tau(s)ds) \end{array}\right);\ \Omega = \sqrt{\theta_1^2+\theta_2^2},$$

$$\mathbf{B}_{\gamma_1} = \frac{1}{\Omega}\left(\frac{(a\theta_1-\tau\theta_2)}{\sqrt{a^2-\tau^2(s)}}, 0, 0\right). \tag{3.34}$$

From the aforementioned calculations, the curvatures of $\gamma_1(s_1)$ are

$$\kappa_{\gamma_1} = \frac{\Omega}{(a^2-\tau^2(s))^2}, \tau_{\gamma_1} = 0, \tag{3.35}$$

where

$$\theta_1(s) = \left((a\tau)\left(a^2-\tau^2\right) + \dot{\tau}\tau^2\right), \theta_2(s) = \left(a\dot{\tau}\tau - \tau^2\left(a^2-\tau^2\right)\right). \tag{3.36}$$

As above, we can do similar calculations for **TB** and **TNB**-Smarandache curves of γ. In all cases, the curvatures are non-constants (they depend on the torsion of the Salkowski curve) and the torions are vanished. Thus, this completes the proof. ■

Remark 3.2 *If $r(s)$ is a family of Anti-Salkowski curves $\delta(s)$ in $G^3(\tau = b = const., \kappa = \kappa(s))$, which can be written as*

$$\delta(s) = \left(s, \int\left[\int\kappa(s)\cos(bs)ds\right]ds, \int\left[\int\kappa(s)\sin(bs)ds\right]ds\right), \tag{3.37}$$

then **TN**, **TB** *and* **TNB**-*Smarandache curves of δ are also plane curves with* non-constant *curvatures.*

4 Spherical indicatrices of a general helix in G^3

In what follows, we investigate the **spherical indicatrix** of the tangent of the helix $\beta(s)$. By differentiating (3.26) with respect to s, we have the tangent spherical curve as follows:

$$\Gamma_{\mathbf{T}} = \beta_{\mathbf{T}}(s_{\mathbf{T}}) = \left(1, \frac{1}{m}\sin(m\int\kappa(s)ds), \frac{-1}{m}\cos(m\int(\kappa(s)ds)\right). \tag{4.1}$$

The Frenet vectors of $\beta_{\mathbf{T}}(s_{\mathbf{T}})$ are

$$\begin{aligned} \mathbf{T}_{\mathbf{T}}(s_{\mathbf{T}}) &= \left(0, \cos(m\int\kappa(s)ds), \sin(m\int\kappa(s)ds)\right), \\ \mathbf{N}_{\mathbf{T}}(s_{\mathbf{T}}) &= \left(0, -\sin(m\int\kappa(s)ds), \cos(m\int\kappa(s)ds)\right), \\ \mathbf{B}_{\mathbf{T}}(s_{\mathbf{T}}) &= (1,0,0), \end{aligned} \tag{4.2}$$

and its curvatures are given by

$$\kappa_{\mathbf{T}}(s_{\mathbf{T}}) = m, \quad \tau_{\mathbf{T}}(s_{\mathbf{T}}) = 0. \tag{4.3}$$

Theorem 4.1 *Let $\beta = \beta(s)$ be a helix in the Galilean space G^3 with $\kappa(s) \neq 0$. The curvatures of the tangent spherical curve $\beta_{\mathbf{T}}(s_{\mathbf{T}})$ of β for each $s_{\mathbf{T}} \in I \subset R$ satisfy the following equalities*

$$\langle \mathbf{T}_{\mathbf{T}}, \beta_{\mathbf{T}} \rangle_{G^3} = 0,\ \langle \mathbf{N}_{\mathbf{T}}, \beta_{\mathbf{T}} \rangle_{G^3} = \frac{-1}{\kappa_{\mathbf{T}}},\ \langle \mathbf{B}_{\mathbf{T}}, \beta_{\mathbf{T}} \rangle_{G^3} = \frac{\kappa'_{\mathbf{T}}}{\kappa\ \kappa^2_{\mathbf{T}}\ \tau_{\mathbf{T}}}.$$

Proof. By assumption we have

$$\langle\ \beta_{\mathbf{T}},\ \beta_{\mathbf{T}} \rangle_{G^3} = d^2, \tag{4.4}$$

for every $s_{\mathbf{T}} \in I \subset R$ and d is the radius of the Galilean sphere S^2_G . By differentiating (4.4) with respect to s, we have

$$\left\langle \frac{d\beta_{\mathbf{T}}}{ds_{\mathbf{T}}} \frac{ds_{\mathbf{T}}}{ds}, \beta_{\mathbf{T}} \right\rangle_{G^3} = 0,$$

where $s_{\mathbf{T}}$ is the arc length of the tangent spherical curve in G^3. So, we get

$$\frac{ds_{\mathbf{T}}}{ds} = \kappa(s)\ ,$$

then

$$\kappa(s)\ \langle \mathbf{T}_{\mathbf{T}}, \beta_{\mathbf{T}} \rangle_{G^3} = 0,$$

$$\langle \mathbf{T}_{\mathbf{T}}, \beta_{\mathbf{T}} \rangle_{G^3} = 0. \tag{4.5}$$

By a new differentiation of (4.5), we find that

$$\kappa_{\mathbf{T}}\ \kappa\ \langle \mathbf{N}_{\mathbf{T}}, \beta_{\mathbf{T}} \rangle_{G^3} + \kappa\ \langle \mathbf{T}_{\mathbf{T}}, \mathbf{T}_{\mathbf{T}} \rangle_{G^3} = 0.$$

From which

$$\langle \mathbf{N}_{\mathbf{T}}, \beta_{\mathbf{T}} \rangle_{G^3} = \frac{-1}{\kappa_{\mathbf{T}}}. \tag{4.6}$$

More differentiation of (4.6) gives

$$-\tau_{\mathbf{T}}\ \kappa\ \langle \mathbf{B}_{\mathbf{T}}, \beta_{\mathbf{T}} \rangle_{G^3} +\ \kappa\ \langle \mathbf{N}_{\mathbf{T}}, \mathbf{T}_{\mathbf{T}} \rangle_{G^3} = \frac{-\kappa_{\mathbf{T}}}{\kappa^2_{\mathbf{T}}},$$

since

$$\langle \mathbf{N}_{\mathbf{T}}, \mathbf{T}_{\mathbf{T}} \rangle_{G^3} = 0, \frac{d\mathbf{N}_{\mathbf{T}}}{ds_{\mathbf{T}}} = -\tau_{\mathbf{T}}\ \mathbf{B}_{\mathbf{T}},$$

then

$$\langle \mathbf{B}_{\mathbf{T}}, \beta_{\mathbf{T}} \rangle_{G^3} = \frac{\kappa_{\mathbf{T}}}{\kappa\ \tau_{\mathbf{T}}\ \kappa^2_{\mathbf{T}}}.$$

Thus, the proof is completed. ■

From aforementioned information, we have the following proposition.

Proposition 4.1 *The Galilean spherical images of a general helix (or circular helix) in the three-dimensional Galilean space are plane curves* ***with constant curvatures.***

Proposition 4.2 *We can also do in similar way the calculations for the other spherical images of* **TN**, **TB** *and* **TNB***-Smarandache curves, we find that they are plane curves* ***with non-constant curvatures****.*

5 Examples

Example 5.1 *Let $\beta(s)$ be a general helix in three-dimensional Galiliean space with $\kappa = \ \tau = \frac{1}{2\sqrt{s}}$, parameterized by*

$$\beta(s) = \left(s, 2\left[\sin\sqrt{s} - \sqrt{s}\cos\sqrt{s}\right], -2\left[\cos\sqrt{s} + \sqrt{s}\sin\sqrt{s}\right]\right)$$

The **TN-** *Smarandache curve of $\beta(s)$ is given by*

$$\beta_1\left(s_1\right) = \left(1, \sin\left(\sqrt{s}\right) + \cos\left(\sqrt{s}\right), -\cos\left(\sqrt{s}\right) + \sin\left(\sqrt{s}\right)\right).$$

Then, the Frenet vectors are

$$\mathbf{T}_1(s_1) = \left(0, \frac{\cos\left(\sqrt{s}\right) - \sin\left(\sqrt{s}\right)}{\sqrt{2}}, \frac{\cos\left(\sqrt{s}\right) + \sin\left(\sqrt{s}\right)}{\sqrt{2}}\right),$$

$$\mathbf{N}_1(s_1) = \left(0, -\frac{\left(\cos\left(\sqrt{s}\right) + \sin\left(\sqrt{s}\right)\right)}{\sqrt{2}}, \frac{\cos\left(\sqrt{s}\right) - \sin\left(\sqrt{s}\right)}{\sqrt{2}}\right),$$

$$\mathbf{B}_1(s_1) = \left(\cos^2\left(\sqrt{s}\right) + \sin^2\left(\sqrt{s}\right), 0, 0\right).$$

The derivatives of these vectors give

$$\kappa_1\left(s_1\right) = \frac{1}{\sqrt{2}}, \ \tau_1\left(s_1\right) = 0.$$

The **TB** *and* **TNB**-*Smarandache curves of $\beta\left(s\right)$ can be calculated as above.*

The tangent spherical image of $\beta(s)$ is defined by

$$\beta_{\mathbf{T}} = \left(1, sin\sqrt{s}, - \ cos\sqrt{s}\right).$$

From which, one can obtain

$$\begin{aligned}
\mathbf{T}_{\mathbf{T}} &= \left(0, cos\sqrt{s}, sin\sqrt{s}\right), \\
\mathbf{N}_{\mathbf{T}} &= \left(0, -sin\sqrt{s}, cos\sqrt{s}\right), \\
\mathbf{B}_{\mathbf{T}} &= \left(\left(sin\sqrt{s}\right)^2 + \left(cos\sqrt{s}\right)^2, 0, 0\right).
\end{aligned}$$

It follows that

$$\kappa_{\mathbf{T}} = 1, \ \tau_{\mathbf{T}} = 0.$$

In an analogous way, one can obtain the normal and binormal spherical images of β and their curvatures. The general helix $\beta(s)$ and its TN-Smarandache curve and tangent spherical image are shown in Figures 1,2,3.

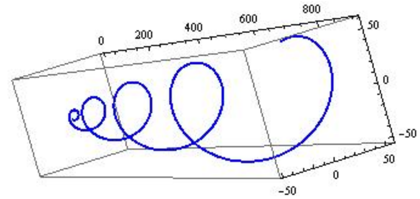

Figure 1: The general helix $\beta(s)$.

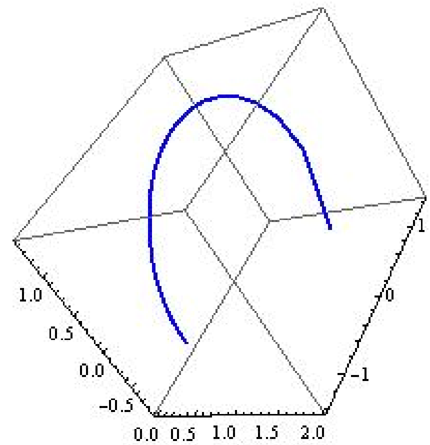

Figure 2: The TN-Smarandache curve of $\beta(s)$.

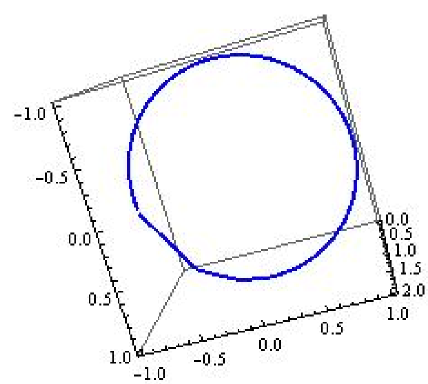

Figure 3: The tangent spherical image of $\beta(s)$.

Example 5.2 *Consider a circular helix $\nu(s)$ in G^3 with non-zero constant curvature and torsion, which is given by*

$$\nu(s) = (s, cos\ s, sin\ s)\,,$$

the representation of its tangent spherical curve is

$$\nu_{\mathbf{T}} = (1, -sin\ s, cos\ s)\,.$$

Then, simple calculations lead to

$$\begin{aligned}
\mathbf{T}_{\mathbf{T}} &= (0, -cos\ s, -sin\ s)\,, \\
\mathbf{N}_{\mathbf{T}} &= (0, sins, -coss)\,, \\
\mathbf{B}_{\mathbf{T}} &= \left((sins)^2 + (coss)^2\,, 0, 0\right).
\end{aligned}$$

By the aid of the derivatives of the above formulas, we obtain the curvatures of $\nu_{\mathbf{T}}$

$$\kappa_{\mathbf{T}} = 1,\ \ \tau_{\mathbf{T}} = 0.$$

The curve ν*and its tangent spherical image are shown in Figures 4,5.*

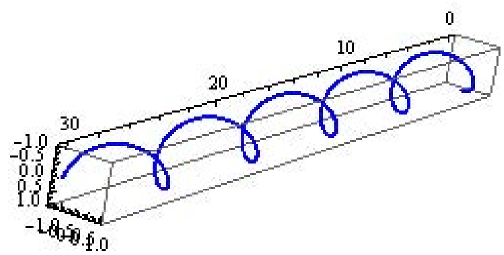

Figure 4: A circular helix $\nu(s)$.

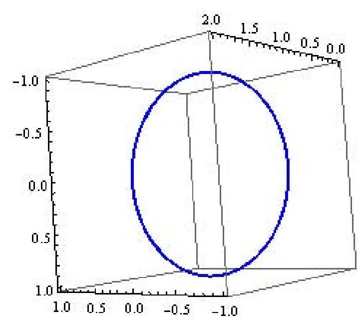

Figure 5: The tangent spherical image of $\nu(s)$.

6 Conclusion

In the three-dimensional Galilean space, Smarandache curves of some special curves such as helix, circular helix, Salkowski and Ant-Salkowski curves are studied. Furthermore, the spherical images of the helix are given. Some interesting results are presented. Finally as an application for this work, some examples are given and plotted.

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