Smarandache Isotopy Theory Of Smarandache: Quasigroups And Loops *†

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Abstract

The concept of Smarandache isotopy is introduced and its study is explored for Smarandache: groupoids, quasigroups and loops just like the study of isotopy theory was carried out for groupoids, quasigroups and loops. The exploration includes: Smarandache; isotopy and isomorphy classes, Smarandache f, g principal isotopes and G-Smarandache loops.

1 Introduction

In 2002, W. B. Vasantha Kandasamy initiated the study of Smarandache loops in her book [12] where she introduced over 75 Smarandache concepts in loops. In her paper [13], she defined a Smarandache loop (S-loop) as a loop with at least a subloop which forms a subgroup under the binary operation of the loop. For more on loops and their properties, readers should check [11], [1], [3], [4], [5] and [12]. In [[12], Page 102], the author introduced Smarandache isotopes of loops particularly Smarandache principal isotopes. She has also introduced the Smarandache concept in some other algebraic structures as [14, 15, 16, 17, 18, 19] account. The present author has contributed to the study of S-quasigroups and S-loops in [6], [7] and [8] while Muktibodh [10] did a study on the first.

In this study, the concept of Smarandache isotopy will be introduced and its study will be explored in Smarandache: groupoids, quasigroups and loops just like the study of isotopy theory was carried out for groupoids, quasigroups and loops as summarized in Bruck [1], Dene and Keedwell [4], Pflugfelder [11].

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2 Definitions and Notations

Definition 2.1 *Let L be a non-empty set. Define a binary operation $(\cdot)$ on L : If $x \cdot y \in L$ $\forall$ $x, y \in L$, $(L, \cdot)$ is called a groupoid. If the system of equations ; $a \cdot x = b$ and $y \cdot a = b$ have unique solutions for x and y respectively, then $(L, \cdot)$ is called a quasigroup. Furthermore, if there exists a unique element $e \in L$ called the identity element such that $\forall$ $x \in L$, $x \cdot e = e \cdot x = x$, $(L, \cdot)$ is called a loop.*

If there exists at least a non-empty and non-trivial subset M of a groupoid(quasigroup or semigroup or loop) L such that $(M, \cdot)$ is a non-trivial subsemigroup(subgroup or subgroup or subgroup) of $(L, \cdot)$, then L is called a Smarandache: groupoid(S-groupoid)$\Big($quasigroup(S-quasigroup) or semigroup(S-semigroup) or loop(S-loop)$\Big)$ with Smarandache: subsemigroup(S-subsemigroup)$\Big($subgroup(S-subgroup) or subgroup(S-subgroup) or subgroup(S-subgroup)$\Big)$ M.

Let $(G, \cdot)$ be a quasigroup(loop). The bijection $L_x : G \to G$ defined as $yL_x = x \cdot y$ $\forall$ $x, y \in G$ is called a left translation(multiplication) of G while the bijection $R_x : G \to G$ defined as $yR_x = y \cdot x$ $\forall$ $x, y \in G$ is called a right translation(multiplication) of G.

The set $SYM(L, \cdot) = SYM(L)$ of all bijections in a groupoid $(L, \cdot)$ forms a group called the permutation(symmetric) group of the groupoid $(L, \cdot)$.

Definition 2.2 *If $(L, \cdot)$ and $(G, \circ)$ are two distinct groupoids, then the triple (U, V, W) : $(L, \cdot) \to (G, \circ)$ such that $U, V, W : L \to G$ are bijections is called an isotopism if and only if*

$$xU \circ yV = (x \cdot y)W \;\forall\; x, y \in L.$$

So we call L and G groupoid isotopes. If $L = G$ and $W = I$(identity mapping) then (U, V, I) is called a principal isotopism, so we call G a principal isotope of L. But if in addition G is a quasigroup such that for some $f, g \in G$, $U = R_g$ and $V = L_f$, then (R_g, L_f, I) : $(G, \cdot) \to (G, \circ)$ is called an f, g-principal isotopism while $(G, \cdot)$ and $(G, \circ)$ are called quasigroup isotopes.

If $U = V = W$, then U is called an isomorphism, hence we write $(L, \cdot) \cong (G, \circ)$. A loop $(L, \cdot)$ is called a G-loop if and only if $(L, \cdot) \cong (G, \circ)$ for all loop isotopes $(G, \circ)$ of $(L, \cdot)$.

Now, if $(L, \cdot)$ and $(G, \circ)$ are S-groupoids with S-subsemigroups L' and G' respectively such that $(G')A = L'$, where $A \in \{U, V, W\}$, then the isotopism $(U, V, W) : (L, \cdot) \to (G, \circ)$ is called a Smarandache isotopism(S-isotopism). Consequently, if $W = I$ the triple (U, V, I) is called a Smarandache principal isotopism. But if in addition G is a S-quasigroup with S-subgroup H' such that for some $f, g \in H$, $U = R_g$ and $V = L_f$, and (R_g, L_f, I) : $(G, \cdot) \to (G, \circ)$ is an isotopism, then the triple is called a Smarandache f, g-principal isotopism while f and g are called Smarandache elements(S-elements).

Thus, if $U = V = W$, then U is called a Smarandache isomorphism, hence we write $(L, \cdot) \succsim (G, \circ)$. An S-loop $(L, \cdot)$ is called a G-Smarandache loop(GS-loop) if and only if $(L, \cdot) \succsim (G, \circ)$ for all loop isotopes(or particularly all S-loop isotopes) $(G, \circ)$ of $(L, \cdot)$.

Example 2.1 *The systems $(L, \cdot)$ and $(L, *)$, $L = \{0, 1, 2, 3, 4\}$ with the multiplication tables below are S-quasigroups with S-subgroups $(L', \cdot)$ and $(L'', *)$ respectively, $L' = \{0, 1\}$ and $L'' = \{1, 2\}$. $(L, \cdot)$ is taken from Example 2.2 of [10]. The triple (U, V, W) such that*

$$U = \begin{pmatrix} 0 & 1 & 2 & 3 & 4 \\ 1 & 2 & 3 & 4 & 0 \end{pmatrix},\ V = \begin{pmatrix} 0 & 1 & 2 & 3 & 4 \\ 1 & 2 & 4 & 0 & 3 \end{pmatrix} \text{ and } W = \begin{pmatrix} 0 & 1 & 2 & 3 & 4 \\ 1 & 2 & 0 & 4 & 3 \end{pmatrix}$$

are permutations on L, is an S-isotopism of $(L, \cdot)$ onto $(L, *)$. Notice that $A(L') = L''$ for all $A \in \{U, V, W\}$ and $U, V, W : L' \to L''$ are all bijcetions.

$\cdot$	0	1	2	3	4
0	0	1	3	4	2
1	1	0	2	3	4
2	3	4	1	2	0
3	4	2	0	1	3
4	2	3	4	0	1

$*$	0	1	2	3	4
0	1	0	4	2	3
1	3	1	2	0	4
2	4	2	1	3	0
3	0	4	3	1	2
4	2	3	0	4	1

Example 2.2 *According to Example 4.2.2 of [15], the system $(\mathbb{Z}_6, \times_6)$ i.e the set $L = \mathbb{Z}_6$ under multiplication modulo 6 is an S-semigroup with S-subgroups $(L', \times_6)$ and $(L'', \times_6)$, $L' = \{2, 4\}$ and $L'' = \{1, 5\}$. This can be deduced from its multiplication table, below. The triple (U, V, W) such that*

$$U = \begin{pmatrix} 0 & 1 & 2 & 3 & 4 & 5 \\ 4 & 3 & 5 & 1 & 2 & 0 \end{pmatrix},\ V = \begin{pmatrix} 0 & 1 & 2 & 3 & 4 & 5 \\ 1 & 3 & 2 & 4 & 5 & 0 \end{pmatrix} \text{ and } W = \begin{pmatrix} 0 & 1 & 2 & 3 & 4 & 5 \\ 1 & 0 & 5 & 4 & 2 & 3 \end{pmatrix}$$

are permutations on L, is an S-isotopism of $(\mathbb{Z}_6, \times_6)$ unto an S-semigroup $(\mathbb{Z}_6, *)$ with S-subgroups $(L''', *)$ and $(L'''', *)$, $L''' = \{2, 5\}$ and $L'''' = \{0, 3\}$ as shown in the second table below. Notice that $A(L') = L'''$ and $A(L'') = L''''$ for all $A \in \{U, V, W\}$ and $U, V, W : L' \to L'''$ and $U, V, W : L'' \to L''''$ are all bijcetions.

$\times_6$	0	1	2	3	4	5
0	0	0	0	0	0	0
1	0	1	2	3	4	5
2	0	2	4	0	2	4
3	0	3	0	3	0	3
4	0	4	2	0	4	2
5	0	5	4	3	2	1

$*$	0	1	2	3	4	5
0	0	1	2	3	4	5
1	4	1	1	4	4	1
2	5	1	5	2	1	2
3	3	1	5	0	4	2
4	1	1	1	1	1	1
5	2	1	2	5	1	5

Remark 2.1 *Taking careful look at Definition 2.2 and comparing it with [Definition 4.4.1,[12]], it will be observed that the author did not allow the component bijections U,V and W in (U, V, W) to act on the whole S-loop L but only on the S-subloop(S-subgroup) L'. We feel this is necessary to adjust here so that the set $L - L'$ is not out of the study. Apart from this, our adjustment here will allow the study of Smarandache isotopy to be*

explorable. Therefore, the S-isotopism and S-isomorphism here are clearly special types of relations(isotopism and isomorphism) on the whole domain into the whole co-domain but those of Vasantha Kandasamy [12] only take care of the structure of the elements in the S-subloop and not the S-loop. Nevertheless, we do not fault her study for we think she defined them to apply them to some life problems as an applied algebraist.

3 Smarandache Isotopy and Isomorphy Classes

Theorem 3.1 *Let $\mathfrak{G} = \left\{ (G_\omega, \circ_\omega) \right\}_{\omega \in \Omega}$ be a set of distinct S-groupoids with a corresponding set of S-subsemigroups $\mathfrak{H} = \left\{ (H_\omega, \circ_\omega) \right\}_{\omega \in \Omega}$. Define a relation $\sim$ on $\mathfrak{G}$ such that for all $(G_{\omega_i}, \circ_{\omega_i})$, $(G_{\omega_j}, \circ_{\omega_j}) \in \mathfrak{G}$, where $\omega_i, \omega_j \in \Omega$,*

$$(G_{\omega_i}, \circ_{\omega_i}) \sim (G_{\omega_j}, \circ_{\omega_j}) \Longleftrightarrow (G_{\omega_i}, \circ_{\omega_i}) \text{ and } (G_{\omega_j}, \circ_{\omega_j}) \text{ are S-isotopic.}$$

Then $\sim$ is an equivalence relation on $\mathfrak{G}$.

Proof
Let $(G_{\omega_i}, \circ_{\omega_i})$, $(G_{\omega_j}, \circ_{\omega_j})$, $(G_{\omega_k}, \circ_{\omega_k}), \in \mathfrak{G}$, where $\omega_i, \omega_j, \omega_k \in \Omega$.

Reflexivity If $I : G_{\omega_i} \to G_{\omega_i}$ is the identity mapping, then

$$xI \circ_{\omega_i} yI = (x \circ_{\omega_i} y)I \ \forall \ x, y \in G_{\omega_i} \Longrightarrow \text{ the triple } (I, I, I) \ : \ (G_{\omega_i}, \circ_{\omega_i}) \to (G_{\omega_i}, \circ_{\omega_i})$$

is an S-isotopism since $(H_{\omega_i})I = H_{\omega_i} \ \forall \ \omega_i \in \Omega$. In fact, it can be simply deduced that every S-groupoid is S-isomorphic to itself.

Symmetry Let $(G_{\omega_i}, \circ_{\omega_i}) \sim (G_{\omega_j}, \circ_{\omega_j})$. Then there exist bijections

$$U, V, W \ : \ (G_{\omega_i}, \circ_{\omega_i}) \longrightarrow (G_{\omega_j}, \circ_{\omega_j}) \text{ such that } (H_{\omega_i})A = H_{\omega_j} \ \forall \ A \in \{U, V, W\}$$

so that the triple

$$\alpha = (U, V, W) \ : \ (G_{\omega_i}, \circ_{\omega_i}) \longrightarrow (G_{\omega_j}, \circ_{\omega_j})$$

is an isotopism. Since each of U, V, W is bijective, then their inverses

$$U^{-1}, V^{-1}, W^{-1} \ : \ (G_{\omega_j}, \circ_{\omega_j}) \longrightarrow (G_{\omega_i}, \circ_{\omega_i})$$

are bijective. In fact, $(H_{\omega_j})A^{-1} = H_{\omega_i} \ \forall \ A \in \{U, V, W\}$ since A is bijective so that the triple

$$\alpha^{-1} = (U^{-1}, V^{-1}, W^{-1}) \ : \ (G_{\omega_j}, \circ_{\omega_j}) \longrightarrow (G_{\omega_i}, \circ_{\omega_i})$$

is an isotopism. Thus, $(G_{\omega_j}, \circ_{\omega_j}) \sim (G_{\omega_i}, \circ_{\omega_i})$.

Transitivity Let $\big(G_{\omega_i},\circ_{\omega_i}\big)\sim\big(G_{\omega_j},\circ_{\omega_j}\big)$ and $\big(G_{\omega_j},\circ_{\omega_j}\big)\sim\big(G_{\omega_k},\circ_{\omega_k}\big)$. Then there exist bijections

$$U_1,V_1,W_1\ :\ \big(G_{\omega_i},\circ_{\omega_i}\big)\longrightarrow\big(G_{\omega_j},\circ_{\omega_j}\big)\text{ and }U_2,V_2,W_2\ :\ \big(G_{\omega_j},\circ_{\omega_j}\big)\longrightarrow\big(G_{\omega_k},\circ_{\omega_k}\big)$$

$$\text{such that }\big(H_{\omega_i}\big)A=H_{\omega_j}\ \forall\ A\in\{U_1,V_1,W_1\}$$
$$\text{and }\big(H_{\omega_j}\big)B=H_{\omega_k}\ \forall\ B\in\{U_2,V_2,W_2\}\text{ so that the triples}$$
$$\alpha_1=(U_1,V_1,W_1)\ :\ \big(G_{\omega_i},\circ_{\omega_i}\big)\longrightarrow\big(G_{\omega_j},\circ_{\omega_j}\big)\text{ and}$$
$$\alpha_2=(U_2,V_2,W_2)\ :\ \big(G_{\omega_j},\circ_{\omega_j}\big)\longrightarrow\big(G_{\omega_k},\circ_{\omega_k}\big)$$

are isotopisms. Since each of U_i,V_i,W_i, $i=1,2$, is bijective, then

$$U_3=U_1U_2,V_3=V_1V_2,W_3=W_1W_2\ :\ \big(G_{\omega_i},\circ_{\omega_i}\big)\longrightarrow\big(G_{\omega_k},\circ_{\omega_k}\big)$$

are bijections such that $\big(H_{\omega_i}\big)A_3=\big(H_{\omega_i}\big)A_1A_2=\big(H_{\omega_j}\big)A_2=H_{\omega_k}$ so that the triple

$$\alpha_3=\alpha_1\alpha_2=(U_3,V_3,W_3)\ :\ \big(G_{\omega_i},\circ_{\omega_i}\big)\longrightarrow\big(G_{\omega_k},\circ_{\omega_k}\big)$$

is an isotopism. Thus, $\big(G_{\omega_i},\circ_{\omega_i}\big)\sim\big(G_{\omega_k},\circ_{\omega_k}\big)$.

Remark 3.1 *As a follow up to Theorem 3.1, the elements of the set* $\mathfrak{G}/\sim$ *will be referred to as Smarandache isotopy classes(S-isotopy classes). Similarly, if* $\sim$ *meant "S-isomorphism" in Theorem 3.1, then the elements of* $\mathfrak{G}/\sim$ *will be referred to as Smarandache isomorphy classes(S-isomorphy classes). Just like isotopy has an advantage over isomorphy in the classification of loops, so also S-isotopy will have advantage over S-isomorphy in the classification of S-loops.*

Corollary 3.1 *Let* $\mathcal{L}_n$, $\mathcal{SL}_n$ *and* $\mathcal{NSL}_n$ *be the sets of; all finite loops of order* n*; all finite S-loops of order* n *and all finite non S-loops of order* n *respectively.*

1. *If* $\mathcal{A}_i^n$ *and* $\mathcal{B}_i^n$ *represent the isomorphy class of* $\mathcal{L}_n$ *and the S-isomorphy class of* $\mathcal{SL}_n$ *respectively, then*

   (a) $|\mathcal{SL}_n|+|\mathcal{NSL}_n|=|\mathcal{L}_n|$;
   - **(i)** $|\mathcal{SL}_5|+|\mathcal{NSL}_5|=56$,
   - **(ii)** $|\mathcal{SL}_6|+|\mathcal{NSL}_6|=9,408$ *and*
   - **(iii)** $|\mathcal{SL}_7|+|\mathcal{NSL}_7|=16,942,080$.

   (b) $|\mathcal{NSL}_n|=\sum_{i=1}|\mathcal{A}_i^n|-\sum_{i=1}|\mathcal{B}_i^n|$;
   - **(i)** $|\mathcal{NSL}_5|=\sum_{i=1}^{6}|\mathcal{A}_i^5|-\sum_{i=1}|\mathcal{B}_i^5|$,
   - **(ii)** $|\mathcal{NSL}_6|=\sum_{i=1}^{109}|\mathcal{A}_i^6|-\sum_{i=1}|\mathcal{B}_i^6|$ *and*
   - **(iii)** $|\mathcal{NSL}_7|=\sum_{i=1}^{23,746}|\mathcal{A}_i^7|-\sum_{i=1}|\mathcal{B}_i^7|$.

2. *If $\mathfrak{A}_i^n$ and $\mathfrak{B}_i^n$ represent the isotopy class of $\mathcal{L}_n$ and the S-isotopy class of $\mathcal{SL}_n$ respectively, then*

$$|\mathcal{NSL}_n| = \sum_{i=1} |\mathfrak{A}_i^n| - \sum_{i=1} |\mathfrak{B}_i^n|;$$

(i) $|\mathcal{NSL}_5| = \sum_{i=1}^{2} |\mathfrak{A}_i^5| - \sum_{i=1} |\mathfrak{B}_i^5|$,

(ii) $|\mathcal{NSL}_6| = \sum_{i=1}^{22} |\mathfrak{A}_i^6| - \sum_{i=1} |\mathfrak{B}_i^6|$ *and*

(iii) $|\mathcal{NSL}_7| = \sum_{i=1}^{564} |\mathfrak{A}_i^7| - \sum_{i=1} |\mathfrak{B}_i^7|$.

Proof

An S-loop is an S-groupoid. Thus by Theorem 3.1, we have S-isomorphy classes and S-isotopy classes. Recall that $|\mathcal{L}_n| = |\mathcal{SL}_n| + |\mathcal{NSL}_n| - |\mathcal{SL}_n \bigcap \mathcal{NSL}_n|$ but $\mathcal{SL}_n \bigcap \mathcal{NSL}_n = \emptyset$ so $|\mathcal{L}_n| = |\mathcal{SL}_n| + |\mathcal{NSL}_n|$. As stated and shown in [11], [5], [2] and [9], the facts in Table 1 are true where n is the order of a finite loop. Hence the claims follow.

n	5	6	7
$\lvert\mathcal{L}_n\rvert$	56	9, 408	16, 942, 080
$\{\mathcal{A}_i^n\}_{i=1}^k$	$k = 6$	$k = 109$	$k = 23,746$
$\{\mathfrak{A}_i^n\}_{i=1}^m$	$m = 2$	$m = 22$	$m = 564$

Table 1: Enumeration of Isomorphy and Isotopy classes of finite loops of small order

Question 3.1 *How many S-loops are in the family $\mathcal{L}_n$? That is, what is $|\mathcal{SL}_n|$ or $|\mathcal{NSL}_n|$.*

Theorem 3.2 *Let $(G, \cdot)$ be a finite S-groupoid of order n with a finite S-subsemigroup $(H, \cdot)$ of order m. Also, let*

$$\mathcal{ISOT}(G, \cdot),\ \mathcal{SISOT}(G, \cdot) \text{ and } \mathcal{NSISOT}(G, \cdot)$$

be the sets of all isotopisms, S-isotopisms and non S-isotopisms of $(G, \cdot)$. Then,

$$\mathcal{ISOT}(G, \cdot) \text{ is a group and } \mathcal{SISOT}(G, \cdot) \leq \mathcal{ISOT}(G, \cdot).$$

Furthermore:

1. $|\mathcal{ISOT}(G, \cdot)| = (n!)^3$;
2. $|\mathcal{SISOT}(G, \cdot)| = (m!)^3$;
3. $|\mathcal{NSISOT}(G, \cdot)| = (n!)^3 - (m!)^3$.

Proof

1. This has been shown to be true in [Theorem 4.1.1, [4]].

2. An S-isotopism is an isotopism. So, $\mathcal{SISOT}(G,\cdot) \subset \mathcal{ISOT}(G,\cdot)$. Thus, we need to just verify the axioms of a group to show that $\mathcal{SISOT}(G,\cdot) \leq \mathcal{ISOT}(G,\cdot)$. These can be done using the proofs of reflexivity, symmetry and transitivity in Theorem 3.1 as guides. For all triples

$$\alpha \in \mathcal{SISOT}(G,\cdot) \text{ such that } \alpha = (U,V,W)\ :\ (G,\cdot) \longrightarrow (G,\circ),$$

where $(G,\cdot)$ and $(G,\circ)$ are S-groupoids with S-subgroups $(H,\cdot)$ and $(K,\circ)$ respectively, we can set

$$U' := U|_H,\ V' := V|_H \text{ and } W' := W|_H \text{ since } A(H) = K\ \forall\ A \in \{U,V,W\},$$

so that $\mathcal{SISOT}(H,\cdot) = \{(U',V',W')\}$. This is possible because of the following arguments.

Let

$$X = \Big\{ f' := f|_H \ \Big|\ f : G \longrightarrow G,\ f : H \longrightarrow K \text{ is bijective and } f(H) = K \Big\}.$$

Let

$$SYM(H,K) = \{\text{bijections from } H \text{ unto } K\}.$$

By definition, it is easy to see that $X \subseteq SYM(H,K)$. Now, for all $U \in SYM(H,K)$, define $U\ :\ H^c \longrightarrow K^c$ so that $U\ :\ G \longrightarrow G$ is a bijection since $|H| = |K|$ implies $|H^c| = |K^c|$. Thus, $SYM(H,K) \subseteq X$ so that $SYM(H,K) = X$.

Given that $|H| = m$, then it follows from (1) that

$$|\mathcal{ISOT}(H,\cdot)| = (m!)^3 \text{ so that } |\mathcal{SISOT}(G,\cdot)| = (m!)^3 \text{ since } SYM(H,K) = X.$$

3. $$\mathcal{NSISOT}(G,\cdot) = \big(\mathcal{SISOT}(G,\cdot)\big)^c.$$

So, the identity isotopism

$$(I,I,I) \notin \mathcal{NSISOT}(G,\cdot),\ \text{hence } \mathcal{NSISOT}(G,\cdot) \nleq \mathcal{ISOT}(G,\cdot).$$

Furthermore,

$$|\mathcal{NSISOT}(G,\cdot)| = (n!)^3 - (m!)^3.$$

Corollary 3.2 *Let $(G,\cdot)$ be a finite S-groupoid of order n with an S-subsemigroup $(H,\cdot)$. If $\mathcal{ISOT}(G,\cdot)$ is the group of all isotopisms of $(G,\cdot)$ and S_n is the symmetric group of degree n, then*

$$\mathcal{ISOT}(G,\cdot) \gtrsim S_n \times S_n \times S_n.$$

Proof
As concluded in [Corollary 1, [4]], $\mathcal{ISOT}(G,\cdot) \cong S_n \times S_n \times S_n$. Let $\mathcal{PISOT}(G,\cdot)$ be the set of all principal isotopisms on $(G,\cdot)$. $\mathcal{PISOT}(G,\cdot)$ is an S-subgroup in $\mathcal{ISOT}(G,\cdot)$ while $S_n \times S_n \times \{I\}$ is an S-subgroup in $S_n \times S_n \times S_n$. If

$$\Upsilon \ : \ \mathcal{ISOT}(G,\cdot) \longrightarrow S_n \times S_n \times S_n \text{ is defined as}$$

$$\Upsilon\big((A,B,I)\big) = < A,B,I > \ \forall \ (A,B,I) \in \mathcal{ISOT}(G,\cdot),$$

then

$$\Upsilon\Big(\mathcal{PISOT}(G,\cdot)\Big) = S_n \times S_n \times \{I\}. \ \therefore \ \mathcal{ISOT}(G,\cdot) \succsim S_n \times S_n \times S_n.$$

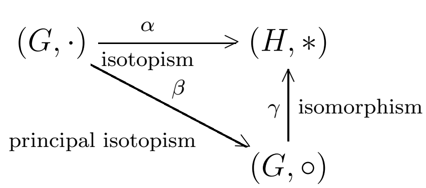
4 Smarandache f,g-Isotopes of Smarandache Loops

Theorem 4.1 *Let $(G,\cdot)$ and $(H,*)$ be S-groupoids. If $(G,\cdot)$ and $(H,*)$ are S-isotopic, then $(H,*)$ is S-isomorphic to some Smarandache principal isotope $(G,\circ)$ of $(G,\cdot)$.*

Proof
Since $(G,\cdot)$ and $(H,*)$ are S-isotopic S-groupoids with S-subsemigroups $(G_1,\cdot)$ and $(H_1,*)$, then there exist bijections $U,V,W \ : \ (G,\cdot) \to (H,*)$ such that the triple $\alpha = (U,V,W) \ : \ (G,\cdot) \to (H,*)$ is an isotopism and $\big(G_1\big)A = H_1 \ \forall \ A \in \{U,V,W\}$. To prove the claim of this theorem, it suffices to produce a closed binary operation '$*$' on G, bijections $X,Y \ : \ G \to G$, and bijection $Z \ : \ G \to H$ so that

- the triple $\beta = (X,Y,I) \ : \ (G,\cdot) \to (G,\circ)$ is a Smarandache principal isotopism and
- $Z \ : \ (G,\circ) \to (H,*)$ is an S-isomorphism or the triple $\gamma = (Z,Z,Z) \ : \ (G,\circ) \to (H,*)$ is an S-isotopism.

Thus, we need $(G,\circ)$ so that the commutative diagram below is true:



because following the proof of transitivity in Theorem 3.1, $\alpha = \beta\gamma$ which implies $(U,V,W) = (XZ,YZ,Z)$ and so we can make the choices; $Z = W$, $Y = VW^{-1}$, and $X = UW^{-1}$ and consequently,

$$x \cdot y = xUW^{-1} \circ VW^{-1} \iff x \circ y = xWU^{-1} \cdot yWV^{-1} \ \forall \ x,y \in G.$$

Hence, $(G,\circ)$ is a groupoid principal isotope of $(G,\cdot)$ and $(H,*)$ is an isomorph of $(G,\circ)$. It remains to show that these two relationships are Smarandache.

Note that $\big((H_1)Z^{-1},\circ\big) = (G_1,\circ)$ is a non-trivial subsemigroup in $(G,\circ)$. Thus, $(G,\circ)$ is an S-groupoid. So $(G,\circ) \succsim (H,*)$. $(G,\cdot)$ and $(G,\circ)$ are Smarandache principal isotopes because $(G_1)UW^{-1} = (H_1)W^{-1} = (H_1)Z^{-1} = G_1$ and $(G_1)VW^{-1} = (H_1)W^{-1} = (H_1)Z^{-1} = G_1$.

Corollary 4.1 *Let $(G, \cdot)$ be an S-groupoid with an arbitrary groupoid isotope $(H, *)$. Any such groupoid $(H, *)$ is an S-groupoid if and only if all the principal isotopes of $(G, \cdot)$ are S-groupoids.*

Proof
By classical result in principal isotopy [[11], III.1.4 Theorem], if $(G, \cdot)$ and $(H, *)$ are isotopic groupoids, then $(H, *)$ is isomorphic to some principal isotope $(G, \circ)$ of $(G, \cdot)$. Assuming $(H, *)$ is an S-groupoid then since $(H, *) \cong (G, \circ)$, $(G, \circ)$ is an S-groupoid. Conversely, let us assume all the principal isotopes of $(G, \cdot)$ are S-groupoids. Since $(H, *) \cong (G, \circ)$, then $(H, *)$ is an S-groupoid.

Theorem 4.2 *Let $(G, \cdot)$ be an S-quasigroup. If $(H, *)$ is an S-loop which is S-isotopic to $(G, \cdot)$, then there exist S-elements f and g so that $(H, *)$ is S-isomorphic to a Smarandache f, g principal isotope $(G, \circ)$ of $(G, \cdot)$.*

Proof
An S-quasigroup and an S-loop are S-groupoids. So by Theorem 4.1, $(H, *)$ is S-isomorphic to a Smarandache principal isotope $(G, \circ)$ of $(G, \cdot)$. Let $\alpha = (U, V, I)$ be the Smarandache principal isotopism of $(G, \cdot)$ onto $(G, \circ)$. Since $(H, *)$ is a S-loop and $(G, \circ) \succsim (H, *)$ implies that $(G, \circ) \cong (H, *)$, then $(G, \circ)$ is necessarily an S-loop and consequently, $(G, \circ)$ has a two-sided identity element say e and an S-subgroup $(G_2, \circ)$. Let $\alpha = (U, V, I)$ be the Smarandache principal isotopism of $(G, \cdot)$ onto $(G, \circ)$. Then,

$$xU \circ yV = x \cdot y \ \forall \ x, y \in G \iff x \circ y = xU^{-1} \cdot yV^{-1} \ \forall \ x, y \in G.$$

So,

$$y = e \circ y = eU^{-1} \cdot yV^{-1} = yV^{-1}L_{eU^{-1}} \ \forall \ y \in G \text{ and } x = x \circ e = xU^{-1} \cdot eV^{-1} = xU^{-1}R_{eV^{-1}} \ \forall \ x \in G.$$

Assign $f = eU^{-1}, g = eV^{-1} \in G_2$. This assignments are well defined and hence $V = L_f$ and $U = R_g$. So that $\alpha = (R_g, L_f, I)$ is a Smarandache f, g principal isotopism of $(G, \circ)$ onto $(G, \cdot)$. This completes the proof.

Corollary 4.2 *Let $(G, \cdot)$ be an S-quasigroup(S-loop) with an arbitrary groupoid isotope $(H, *)$. Any such groupoid $(H, *)$ is an S-quasigroup(S-loop) if and only if all the principal isotopes of $(G, \cdot)$ are S-quasigroups(S-loops).*

Proof
This follows immediately from Corollary 4.1 since an S-quasigroup and an S-loop are S-groupoids.

Corollary 4.3 *If $(G, \cdot)$ and $(H, *)$ are S-loops which are S-isotopic, then there exist S-elements f and g so that $(H, *)$ is S-isomorphic to a Smarandache f, g principal isotope $(G, \circ)$ of $(G, \cdot)$.*

Proof
An S-loop is an S-quasigroup. So the claim follows from Theorem 4.2.

5 G-Smarandache Loops

Lemma 5.1 *Let $(G,\cdot)$ and $(H,*)$ be S-isotopic S-loops. If $(G,\cdot)$ is a group, then $(G,\cdot)$ and $(H,*)$ are S-isomorphic groups.*

Proof
By Corollary 4.3, there exist S-elements f and g in $(G,\cdot)$ so that $(H,*) \succsim (G,\circ)$ such that $(G,\circ)$ is a Smarandache f,g principal isotope of $(G,\cdot)$.

Let us set the mapping $\psi := R_{f\cdot g} = R_{fg} : G \to G$. This mapping is bijective. Now, let us consider when $\psi := R_{fg} : (G,\cdot) \to (G,\circ)$. Since $(G,\cdot)$ is associative and $x \circ y = xR_g^{-1} \cdot yL_f^{-1}\ \forall\ x,y \in G$, the following arguments are true.

$x\psi \circ y\psi = x\psi R_g^{-1} \cdot y\psi L_f^{-1} = xR_{fg}R_g^{-1} \cdot yR_{fg}L_f^{-1} = x \cdot fg \cdot g^{-1} \cdot f^{-1} \cdot y \cdot fg = x \cdot y \cdot fg = (x \cdot y)R_{fg} = (x \cdot y)\psi\ \forall\ x,y \in G$. So, $(G,\cdot) \cong (G,\circ)$. Thus, $(G,\circ)$ is a group. If $(G_1,\cdot)$ and $(G_1,\circ)$ are the S-subgroups in $(G,\cdot)$ and $(G,\circ)$, then $\big((G_1,\cdot)\big)R_{fg} = (G_1,\circ)$. Hence, $(G,\cdot) \succsim (G,\circ)$.

$\therefore\ (G,\cdot) \succsim (H,*)$ and $(H,*)$ is a group.

Corollary 5.1 *Every group which is an S-loop is a GS-loop.*

Proof
This follows immediately from Lemma 5.1 and the fact that a group is a G-loop.

Corollary 5.2 *An S-loop is S-isomorphic to all its S-loop S-isotopes if and only if it is S-isomorphic to all its Smarandache f,g principal isotopes.*

Proof
Let $(G,\cdot)$ be an S-loop with arbitrary S-isotope $(H,*)$. Let us assume that $(G,\cdot) \succsim (H,*)$. From Corollary 4.3, for any arbitrary S-isotope $(H,*)$ of $(G,\cdot)$, there exists a Smarandache f,g principal isotope $(G,\circ)$ of $(G,\cdot)$ such that $(H,*) \succsim (G,\circ)$. So, $(G,\cdot) \succsim (G,\circ)$.

Conversely, let $(G,\cdot) \succsim (G,\circ)$, using the fact in Corollary 4.3 again, for any arbitrary S-isotope $(H,*)$ of $(G,\cdot)$, there exists a Smarandache f,g principal isotope $(G,\circ)$ of $(G,\cdot)$ such that $(G,\circ) \succsim (H,*)$. Therefore, $(G,\cdot) \succsim (H,*)$.

Corollary 5.3 *A S-loop is a GS-loop if and only if it is S-isomorphic to all its Smarandache f,g principal isotopes.*

Proof
This follows by the definition of a GS-loop and Corollary 5.2.

References

[1] R. H. Bruck (1966), *A survey of binary systems*, Springer-Verlag, Berlin-Göttingen-Heidelberg, 185pp.

[2] R. E. Cawagas (2000), *Generation of NAFIL loops of small order*, Quasigroups and Related Systems, 7, 1–5.

[3] O. Chein, H. O. Pflugfelder and J. D. H. Smith (1990), *Quasigroups and loops : Theory and applications*, Heldermann Verlag, 568pp.

[4] J. Déne and A. D. Keedwell (1974), *Latin squares and their applications*, the English University press Lts, 549pp.

[5] E. G. Goodaire, E. Jespers and C. P. Milies (1996), *Alternative loop rings*, NHMS(184), Elsevier, 387pp.

[6] T. G. Jaíyéọlá (2006), *An holomorphic study of the Smarandache concept in loops*, Scientia Magna Journal, 2, 1, 1–8.

[7] T. G. Jaíyéọlá (2006), *Parastrophic invariance of Smarandache quasigroups*, Scientia Magna Journal, 2, 3, 48–53.

[8] T. G. Jaíyéọlá (2006), *On the universality of some Smarandache loops of Bol-Moufang type*, Scientia Magna Journal, 2, 4, 45–48.

[9] B. D. McKay, A. Meynert and W. Myrvold (2007), *Small latin squares, quasigroups and loops*, Journal of Combinatorial Designs, 15, 2, 98–119.

[10] A. S. Muktibodh (2006), *Smarandache quasigroups*, Scientia Magna Journal, 2, 1, 13–19.

[11] H. O. Pflugfelder (1990), *Quasigroups and loops : Introduction*, Sigma series in Pure Math. 7, Heldermann Verlag, Berlin, 147pp.

[12] W. B. Vasantha Kandasamy (2002), *Smarandache loops*, Department of Mathematics, Indian Institute of Technology, Madras, India, 128pp.

[13] W. B. Vasantha Kandasamy (2002), *Smarandache Loops*, Smarandache Notions Journal, 13, 252–258.

[14] W. B. Vasantha Kandasamy (2002), *Groupoids and Smarandache Groupoids*, American Research Press Rehoboth, 114pp.

[15] W. B. Vasantha Kandasamy (2002), *Smarandache Semigroups*, American Research Press Rehoboth, 94pp.

[16] W. B. Vasantha Kandasamy (2002), *Smarandache Semirings, Semifields, And Semivector Spaces*, American Research Press Rehoboth, 121pp.

[17] W. B. Vasantha Kandasamy (2003), *Linear Algebra And Smarandache Linear Algebra*, American Research Press, 174pp.

[18] W. B. Vasantha Kandasamy (2003), *Bialgebraic Structures And Smarandache Bialgebraic Structures*, American Research Press Rehoboth, 271pp.

[19] W. B. Vasantha Kandasamy and F. Smarandache (2005), *N-Algebraic Structures And Smarandache N-Algebraic Structures*, Hexis Phoenix, Arizona, 174pp.