Smarandache Isotopy Theory Of Smarandache: Quasigroups And Loops *†

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Abstract

The concept of Smarandache isotopy is introduced and its study is explored for Smarandache: groupoids, quasigroups and loops just like the study of isotopy theory was carried out for groupoids, quasigroups and loops. The exploration includes: Smarandache; isotopy and isomorphy classes, Smarandache f, g principal isotopes and G-Smarandache loops.

1 Introduction

In 2002, W. B. Vasantha Kandasamy initiated the study of Smarandache loops in her book [12] where she introduced over 75 Smarandache concepts in loops. In her paper [13], she defined a Smarandache loop (S-loop) as a loop with at least a subloop which forms a subgroup under the binary operation of the loop. For more on loops and their properties, readers should check [11], [1], [3], [4], [5] and [12]. In [[12], Page 102], the author introduced Smarandache isotopes of loops particularly Smarandache principal isotopes. She has also introduced the Smarandache concept in some other algebraic structures as [14, 15, 16, 17, 18, 19] account. The present author has contributed to the study of S-quasigroups and S-loops in [6], [7] and [8] while Muktibodh [10] did a study on the first.

In this study, the concept of Smarandache isotopy will be introduced and its study will be explored in Smarandache: groupoids, quasigroups and loops just like the study of isotopy theory was carried out for groupoids, quasigroups and loops as summarized in Bruck [1], Dene and Keedwell [4], Pflugfelder [11].

*2000 Mathematics Subject Classification. Primary 20NO5 ; Secondary 08A05.
†**Keywords and Phrases :** Smarandache: groupoids; quasigroups; loops; f,g principal isotopes

2 Definitions and Notations

Definition 2.1 *Let L be a non-empty set. Define a binary operation $(\cdot)$ on L : If $x \cdot y \in L \ \forall \ x, y \in L$, $(L, \cdot)$ is called a groupoid. If the system of equations ; $a \cdot x = b$ and $y \cdot a = b$ have unique solutions for x and y respectively, then $(L, \cdot)$ is called a quasigroup. Furthermore, if there exists a unique element $e \in L$ called the identity element such that $\forall \ x \in L$, $x \cdot e = e \cdot x = x$, $(L, \cdot)$ is called a loop.*

If there exists at least a non-empty and non-trivial subset M of a groupoid(quasigroup or semigroup or loop) L such that $(M, \cdot)$ is a non-trivial subsemigroup(subgroup or subgroup or subgroup) of $(L, \cdot)$, then L is called a Smarandache: groupoid(S-groupoid)$\Big($quasigroup(S-quasigroup) or semigroup(S-semigroup) or loop(S-loop)$\Big)$ with Smarandache: subsemigroup(S-subsemigroup)$\Big($subgroup(S-subgroup) or subgroup(S-subgroup) or subgroup(S-subgroup)$\Big)$ M.

Let $(G, \cdot)$ be a quasigroup(loop). The bijection $L_x : G \to G$ defined as $yL_x = x \cdot y \ \forall \ x, y \in G$ is called a left translation(multiplication) of G while the bijection $R_x : G \to G$ defined as $yR_x = y \cdot x \ \forall \ x, y \in G$ is called a right translation(multiplication) of G.

The set $SYM(L, \cdot) = SYM(L)$ of all bijections in a groupoid $(L, \cdot)$ forms a group called the permutation(symmetric) group of the groupoid $(L, \cdot)$.

Definition 2.2 *If $(L, \cdot)$ and $(G, \circ)$ are two distinct groupoids, then the triple $(U, V, W) : (L, \cdot) \to (G, \circ)$ such that $U, V, W : L \to G$ are bijections is called an isotopism if and only if*

So we call L and G groupoid isotopes. If $L = G$ and $W = I$(identity mapping) then (U, V, I) is called a principal isotopism, so we call G a principal isotope of L. But if in addition G is a quasigroup such that for some $f, g \in G$, $U = R_g$ and $V = L_f$, then $(R_g, L_f, I) : (G, \cdot) \to (G, \circ)$ is called an f, g-principal isotopism while $(G, \cdot)$ and $(G, \circ)$ are called quasigroup isotopes.

If $U = V = W$, then U is called an isomorphism, hence we write $(L, \cdot) \cong (G, \circ)$. A loop $(L, \cdot)$ is called a G-loop if and only if $(L, \cdot) \cong (G, \circ)$ for all loop isotopes $(G, \circ)$ of $(L, \cdot)$.

Now, if $(L, \cdot)$ and $(G, \circ)$ are S-groupoids with S-subsemigroups L' and G' respectively such that $(G')A = L'$, where $A \in \{U, V, W\}$, then the isotopism $(U, V, W) : (L, \cdot) \to (G, \circ)$ is called a Smarandache isotopism(S-isotopism). Consequently, if $W = I$ the triple (U, V, I) is called a Smarandache principal isotopism. But if in addition G is a S-quasigroup with S-subgroup H' such that for some $f, g \in H$, $U = R_g$ and $V = L_f$, and $(R_g, L_f, I) : (G, \cdot) \to (G, \circ)$ is an isotopism, then the triple is called a Smarandache f, g-principal isotopism while f and g are called Smarandache elements(S-elements).

Thus, if $U = V = W$, then U is called a Smarandache isomorphism, hence we write $(L, \cdot) \succsim (G, \circ)$. An S-loop $(L, \cdot)$ is called a G-Smarandache loop(GS-loop) if and only if $(L, \cdot) \succsim (G, \circ)$ for all loop isotopes(or particularly all S-loop isotopes) $(G, \circ)$ of $(L, \cdot)$.

Example 2.1 *The systems $(L, \cdot)$ and $(L, \ast)$, $L = \{0, 1, 2, 3, 4\}$ with the multiplication tables below are S-quasigroups with S-subgroups $(L', \cdot)$ and $(L'', \ast)$ respectively, $L' = \{0, 1\}$ and*

$L'' = \{1, 2\}$. $(L, \cdot)$ is taken from Example 2.2 of [10]. The triple (U, V, W) such that

$$U = \begin{pmatrix} 0 & 1 & 2 & 3 & 4 \\ 1 & 2 & 3 & 4 & 0 \end{pmatrix}, \; V = \begin{pmatrix} 0 & 1 & 2 & 3 & 4 \\ 1 & 2 & 4 & 0 & 3 \end{pmatrix} \text{ and } W = \begin{pmatrix} 0 & 1 & 2 & 3 & 4 \\ 1 & 2 & 0 & 4 & 3 \end{pmatrix}$$

are permutations on L, is an S-isotopism of $(L, \cdot)$ onto $(L, *)$. Notice that $A(L') = L''$ for all $A \in \{U, V, W\}$ and $U, V, W : L' \to L''$ are all bijcetions.

$\cdot$	0	1	2	3	4
0	0	1	3	4	2
1	1	0	2	3	4
2	3	4	1	2	0
3	4	2	0	1	3
4	2	3	4	0	1

$*$	0	1	2	3	4
0	1	0	4	2	3
1	3	1	2	0	4
2	4	2	1	3	0
3	0	4	3	1	2
4	2	3	0	4	1

Example 2.2 *According to Example 4.2.2 of [15], the system $(\mathbb{Z}_6, \times_6)$ i.e the set $L = \mathbb{Z}_6$ under multiplication modulo 6 is an S-semigroup with S-subgroups $(L', \times_6)$ and $(L'', \times_6)$, $L' = \{2, 4\}$ and $L'' = \{1, 5\}$. This can be deduced from its multiplication table, below. The triple (U, V, W) such that*

$$U = \begin{pmatrix} 0 & 1 & 2 & 3 & 4 & 5 \\ 4 & 3 & 5 & 1 & 2 & 0 \end{pmatrix}, \; V = \begin{pmatrix} 0 & 1 & 2 & 3 & 4 & 5 \\ 1 & 3 & 2 & 4 & 5 & 0 \end{pmatrix} \text{ and } W = \begin{pmatrix} 0 & 1 & 2 & 3 & 4 & 5 \\ 1 & 0 & 5 & 4 & 2 & 3 \end{pmatrix}$$

are permutations on L, is an S-isotopism of $(\mathbb{Z}_6, \times_6)$ unto an S-semigroup $(\mathbb{Z}_6, *)$ with S-subgroups $(L''', *)$ and $(L'''', *)$, $L''' = \{2, 5\}$ and $L'''' = \{0, 3\}$ as shown in the second table below. Notice that $A(L') = L'''$ and $A(L'') = L''''$ for all $A \in \{U, V, W\}$ and $U, V, W : L' \to L'''$ and $U, V, W : L'' \to L''''$ are all bijcetions.

$\times_6$	0	1	2	3	4	5
0	0	0	0	0	0	0
1	0	1	2	3	4	5
2	0	2	4	0	2	4
3	0	3	0	3	0	3
4	0	4	2	0	4	2
5	0	5	4	3	2	1

$*$	0	1	2	3	4	5
0	0	1	2	3	4	5
1	4	1	1	4	4	1
2	5	1	5	2	1	2
3	3	1	5	0	4	2
4	1	1	1	1	1	1
5	2	1	2	5	1	5

Remark 2.1 *Taking careful look at Definition 2.2 and comparing it with [Definition 4.4.1,[12]], it will be observed that the author did not allow the component bijections U,V and W in (U, V, W) to act on the whole S-loop L but only on the S-subloop(S-subgroup) L'. We feel this is necessary to adjust here so that the set $L - L'$ is not out of the study. Apart from this, our adjustment here will allow the study of Smarandache isotopy to be explorable. Therefore, the S-isotopism and S-isomorphism here are clearly special types of relations(isotopism and isomorphism) on the whole domain into the whole co-domain but those of Vasantha Kandasamy [12] only take care of the structure of the elements in the S-subloop and not the S-loop. Nevertheless, we do not fault her study for we think she defined them to apply them to some life problems as an applied algebraist.*

3 Smarandache Isotopy and Isomorphy Classes

Theorem 3.1 *Let* $\mathfrak{G} = \Big\{\big(G_\omega, \circ_\omega\big)\Big\}_{\omega\in\Omega}$ *be a set of distinct S-groupoids with a corresponding set of S-subsemigroups* $\mathfrak{H} = \Big\{\big(H_\omega, \circ_\omega\big)\Big\}_{\omega\in\Omega}$*. Define a relation* $\sim$ *on* $\mathfrak{G}$ *such that for all* $\big(G_{\omega_i}, \circ_{\omega_i}\big)$, $\big(G_{\omega_j}, \circ_{\omega_j}\big) \in \mathfrak{G}$*, where* $\omega_i, \omega_j \in \Omega$*,*

$$\big(G_{\omega_i}, \circ_{\omega_i}\big) \sim \big(G_{\omega_j}, \circ_{\omega_j}\big) \Longleftrightarrow \big(G_{\omega_i}, \circ_{\omega_i}\big) \text{ and } \big(G_{\omega_j}, \circ_{\omega_j}\big) \text{ are S-isotopic.}$$

Then $\sim$ *is an equivalence relation on* $\mathfrak{G}$*.*

Proof
Let $\big(G_{\omega_i}, \circ_{\omega_i}\big)$, $\big(G_{\omega_j}, \circ_{\omega_j}\big)$, $\big(G_{\omega_k}, \circ_{\omega_k}\big), \in \mathfrak{G}$, where $\omega_i, \omega_j, \omega_k \in \Omega$.

Reflexivity If $I : G_{\omega_i} \to G_{\omega_i}$ is the identity mapping, then

$$xI \circ_{\omega_i} yI = (x \circ_{\omega_i} y)I \ \forall \ x, y \in G_{\omega_i} \Longrightarrow \text{ the triple } (I, I, I) \ : \ \big(G_{\omega_i}, \circ_{\omega_i}\big) \to \big(G_{\omega_i}, \circ_{\omega_i}\big)$$

is an S-isotopism since $\big(H_{\omega_i}\big)I = H_{\omega_i} \ \forall \ \omega_i \in \Omega$. In fact, it can be simply deduced that every S-groupoid is S-isomorphic to itself.

Symmetry Let $\big(G_{\omega_i}, \circ_{\omega_i}\big) \sim \big(G_{\omega_j}, \circ_{\omega_j}\big)$. Then there exist bijections

$$U, V, W \ : \ \big(G_{\omega_i}, \circ_{\omega_i}\big) \longrightarrow \big(G_{\omega_j}, \circ_{\omega_j}\big) \text{ such that } \big(H_{\omega_i}\big)A = H_{\omega_j} \ \forall \ A \in \{U, V, W\}$$

so that the triple

$$\alpha = (U, V, W) \ : \ \big(G_{\omega_i}, \circ_{\omega_i}\big) \longrightarrow \big(G_{\omega_j}, \circ_{\omega_j}\big)$$

is an isotopism. Since each of U, V, W is bijective, then their inverses

$$U^{-1}, V^{-1}, W^{-1} \ : \ \big(G_{\omega_j}, \circ_{\omega_j}\big) \longrightarrow \big(G_{\omega_i}, \circ_{\omega_i}\big)$$

are bijective. In fact, $\big(H_{\omega_j}\big)A^{-1} = H_{\omega_i} \ \forall \ A \in \{U, V, W\}$ since A is bijective so that the triple

$$\alpha^{-1} = (U^{-1}, V^{-1}, W^{-1}) \ : \ \big(G_{\omega_j}, \circ_{\omega_j}\big) \longrightarrow \big(G_{\omega_i}, \circ_{\omega_i}\big)$$

is an isotopism. Thus, $\big(G_{\omega_j}, \circ_{\omega_j}\big) \sim \big(G_{\omega_i}, \circ_{\omega_i}\big)$.

Transitivity Let $\big(G_{\omega_i}, \circ_{\omega_i}\big) \sim \big(G_{\omega_j}, \circ_{\omega_j}\big)$ and $\big(G_{\omega_j}, \circ_{\omega_j}\big) \sim \big(G_{\omega_k}, \circ_{\omega_k}\big)$. Then there exist bijections

$$U_1, V_1, W_1 \ : \ \big(G_{\omega_i}, \circ_{\omega_i}\big) \longrightarrow \big(G_{\omega_j}, \circ_{\omega_j}\big) \text{ and } U_2, V_2, W_2 \ : \ \big(G_{\omega_j}, \circ_{\omega_j}\big) \longrightarrow \big(G_{\omega_k}, \circ_{\omega_k}\big)$$

$$\text{such that } \big(H_{\omega_i}\big)A = H_{\omega_j} \ \forall \ A \in \{U_1, V_1, W_1\}$$

$$\text{and } \big(H_{\omega_j}\big)B = H_{\omega_k} \ \forall \ B \in \{U_2, V_2, W_2\} \text{ so that the triples}$$

$$\alpha_1 = (U_1, V_1, W_1) \ : \ \big(G_{\omega_i}, \circ_{\omega_i}\big) \longrightarrow \big(G_{\omega_j}, \circ_{\omega_j}\big) \text{ and}$$

$$\alpha_2 = (U_2, V_2, W_2) \ : \ \big(G_{\omega_j}, \circ_{\omega_j}\big) \longrightarrow \big(G_{\omega_k}, \circ_{\omega_k}\big)$$

are isotopisms. Since each of U_i, V_i, W_i, $i = 1, 2$, is bijective, then

$$U_3 = U_1U_2, V_3 = V_1V_2, W_3 = W_1W_2 \ : \ \big(G_{\omega_i}, \circ_{\omega_i}\big) \longrightarrow \big(G_{\omega_k}, \circ_{\omega_k}\big)$$

are bijections such that $\big(H_{\omega_i}\big)A_3 = \big(H_{\omega_i}\big)A_1A_2 = \big(H_{\omega_j}\big)A_2 = H_{\omega_k}$ so that the triple

$$\alpha_3 = \alpha_1\alpha_2 = (U_3, V_3, W_3) \ : \ \big(G_{\omega_i}, \circ_{\omega_i}\big) \longrightarrow \big(G_{\omega_k}, \circ_{\omega_k}\big)$$

is an isotopism. Thus, $\big(G_{\omega_i}, \circ_{\omega_i}\big) \sim \big(G_{\omega_k}, \circ_{\omega_k}\big)$.

Remark 3.1 *As a follow up to Theorem 3.1, the elements of the set $\mathfrak{G}/\sim$ will be referred to as Smarandache isotopy classes(S-isotopy classes). Similarly, if $\sim$ meant "S-isomorphism" in Theorem 3.1, then the elements of $\mathfrak{G}/\sim$ will be referred to as Smarandache isomorphy classes(S-isomorphy classes). Just like isotopy has an advantage over isomorphy in the classification of loops, so also S-isotopy will have advantage over S-isomorphy in the classification of S-loops.*

Corollary 3.1 *Let $\mathcal{L}_n$, $\mathcal{SL}_n$ and $\mathcal{NSL}_n$ be the sets of; all finite loops of order n; all finite S-loops of order n and all finite non S-loops of order n respectively.*

1. *If $\mathcal{A}_i^n$ and $\mathcal{B}_i^n$ represent the isomorphy class of $\mathcal{L}_n$ and the S-isomorphy class of $\mathcal{SL}_n$ respectively, then*

   (a) $|\mathcal{SL}_n| + |\mathcal{NSL}_n| = |\mathcal{L}_n|$;

   - **(i)** $|\mathcal{SL}_5| + |\mathcal{NSL}_5| = 56$,
   - **(ii)** $|\mathcal{SL}_6| + |\mathcal{NSL}_6| = 9,408$ *and*
   - **(iii)** $|\mathcal{SL}_7| + |\mathcal{NSL}_7| = 16,942,080$.

   (b) $|\mathcal{NSL}_n| = \sum_{i=1} |\mathcal{A}_i^n| - \sum_{i=1} |\mathcal{B}_i^n|$;

   - **(i)** $|\mathcal{NSL}_5| = \sum_{i=1}^{6} |\mathcal{A}_i^5| - \sum_{i=1} |\mathcal{B}_i^5|$,
   - **(ii)** $|\mathcal{NSL}_6| = \sum_{i=1}^{109} |\mathcal{A}_i^6| - \sum_{i=1} |\mathcal{B}_i^6|$ *and*
   - **(iii)** $|\mathcal{NSL}_7| = \sum_{i=1}^{23,746} |\mathcal{A}_i^7| - \sum_{i=1} |\mathcal{B}_i^7|$.

2. *If $\mathfrak{A}_i^n$ and $\mathfrak{B}_i^n$ represent the isotopy class of $\mathcal{L}_n$ and the S-isotopy class of $\mathcal{SL}_n$ respectively, then*

$$|\mathcal{NSL}_n| = \sum_{i=1} |\mathfrak{A}_i^n| - \sum_{i=1} |\mathfrak{B}_i^n|;$$

   - **(i)** $|\mathcal{NSL}_5| = \sum_{i=1}^{2} |\mathfrak{A}_i^5| - \sum_{i=1} |\mathfrak{B}_i^5|$,
   - **(ii)** $|\mathcal{NSL}_6| = \sum_{i=1}^{22} |\mathfrak{A}_i^6| - \sum_{i=1} |\mathfrak{B}_i^6|$ *and*
   - **(iii)** $|\mathcal{NSL}_7| = \sum_{i=1}^{564} |\mathfrak{A}_i^7| - \sum_{i=1} |\mathfrak{B}_i^7|$.

Proof
An S-loop is an S-groupoid. Thus by Theorem 3.1, we have S-isomorphy classes and S-isotopy classes. Recall that $|\mathcal{L}_n| = |\mathcal{SL}_n| + |\mathcal{NSL}_n| - |\mathcal{SL}_n \bigcap \mathcal{NSL}_n|$ but $\mathcal{SL}_n \bigcap \mathcal{NSL}_n = \emptyset$ so $|\mathcal{L}_n| = |\mathcal{SL}_n| + |\mathcal{NSL}_n|$. As stated and shown in [11], [5], [2] and [9], the facts in Table 1 are true where n is the order of a finite loop. Hence the claims follow.

Question 3.1 *How many S-loops are in the family $\mathcal{L}_n$? That is, what is $|\mathcal{SL}_n|$ or $|\mathcal{NSL}_n|$.*

Theorem 3.2 *Let $(G,\cdot)$ be a finite S-groupoid of order n with a finite S-subsemigroup $(H,\cdot)$ of order m. Also, let*

$$\mathcal{ISOT}(G,\cdot),\ \mathcal{SISOT}(G,\cdot) \text{ and } \mathcal{NSISOT}(G,\cdot)$$

be the sets of all isotopisms, S-isotopisms and non S-isotopisms of $(G,\cdot)$. Then,

$$\mathcal{ISOT}(G,\cdot) \text{ is a group and } \mathcal{SISOT}(G,\cdot) \leq \mathcal{ISOT}(G,\cdot).$$

Furthermore:

1. $|\mathcal{ISOT}(G,\cdot)| = (n!)^3$;
2. $|\mathcal{SISOT}(G,\cdot)| = (m!)^3$;
3. $|\mathcal{NSISOT}(G,\cdot)| = (n!)^3 - (m!)^3$.

Proof

1. This has been shown to be true in [Theorem 4.1.1, [4]].
2. An S-isotopism is an isotopism. So, $\mathcal{SISOT}(G,\cdot) \subset \mathcal{ISOT}(G,\cdot)$. Thus, we need to just verify the axioms of a group to show that $\mathcal{SISOT}(G,\cdot) \leq \mathcal{ISOT}(G,\cdot)$. These can be done using the proofs of reflexivity, symmetry and transitivity in Theorem 3.1 as guides. For all triples

$$\alpha \in \mathcal{SISOT}(G,\cdot) \text{ such that } \alpha = (U,V,W)\ :\ (G,\cdot) \longrightarrow (G,\circ),$$

where $(G,\cdot)$ and $(G,\circ)$ are S-groupoids with S-subgroups $(H,\cdot)$ and $(K,\circ)$ respectively, we can set

$$U' := U|_H,\ V' := V|_H \text{ and } W' := W|_H \text{ since } A(H) = K\ \forall\ A \in \{U,V,W\},$$

n	5	6	7
$\lvert\mathcal{L}_n\rvert$	56	9, 408	16, 942, 080
$\{\mathcal{A}_i^n\}_{i=1}^k$	$k = 6$	$k = 109$	$k = 23,746$
$\{\mathfrak{A}_i^n\}_{i=1}^m$	$m = 2$	$m = 22$	$m = 564$

Table 1: Enumeration of Isomorphy and Isotopy classes of finite loops of small order

so that $\mathcal{SISOT}(H,\cdot) = \{(U',V',W')\}$. This is possible because of the following arguments.

Let

$$X = \Big\{ f' := f|_H \;\Big|\; f : G \longrightarrow G,\ f : H \longrightarrow K \text{ is bijective and } f(H) = K \Big\}.$$

Let

$$SYM(H,K) = \{\text{bijections from } H \text{ unto } K\}.$$

By definition, it is easy to see that $X \subseteq SYM(H,K)$. Now, for all $U \in SYM(H,K)$, define $U : H^c \longrightarrow K^c$ so that $U : G \longrightarrow G$ is a bijection since $|H| = |K|$ implies $|H^c| = |K^c|$. Thus, $SYM(H,K) \subseteq X$ so that $SYM(H,K) = X$.

Given that $|H| = m$, then it follows from (1) that

$$|\mathcal{ISOT}(H,\cdot)| = (m!)^3 \text{ so that } |\mathcal{SISOT}(G,\cdot)| = (m!)^3 \text{ since } SYM(H,K) = X.$$

3.

$$\mathcal{NSISOT}(G,\cdot) = \big(\mathcal{SISOT}(G,\cdot)\big)^c.$$

So, the identity isotopism

$$(I,I,I) \notin \mathcal{NSISOT}(G,\cdot), \text{ hence } \mathcal{NSISOT}(G,\cdot) \nleq \mathcal{ISOT}(G,\cdot).$$

Furthermore,

$$|\mathcal{NSISOT}(G,\cdot)| = (n!)^3 - (m!)^3.$$

Corollary 3.2 *Let $(G,\cdot)$ be a finite S-groupoid of order n with an S-subsemigroup $(H,\cdot)$. If $\mathcal{ISOT}(G,\cdot)$ is the group of all isotopisms of $(G,\cdot)$ and S_n is the symmetric group of degree n, then*

$$\mathcal{ISOT}(G,\cdot) \succsim S_n \times S_n \times S_n.$$

Proof

As concluded in [Corollary 1, [4]], $\mathcal{ISOT}(G,\cdot) \cong S_n \times S_n \times S_n$. Let $\mathcal{PISOT}(G,\cdot)$ be the set of all principal isotopisms on $(G,\cdot)$. $\mathcal{PISOT}(G,\cdot)$ is an S-subgroup in $\mathcal{ISOT}(G,\cdot)$ while $S_n \times S_n \times \{I\}$ is an S-subgroup in $S_n \times S_n \times S_n$. If

$$\Upsilon : \mathcal{ISOT}(G,\cdot) \longrightarrow S_n \times S_n \times S_n \text{ is defined as}$$

$$\Upsilon\big((A,B,I)\big) = \langle A,B,I \rangle \;\; \forall\; (A,B,I) \in \mathcal{ISOT}(G,\cdot),$$

then

$$\Upsilon\Big(\mathcal{PISOT}(G,\cdot)\Big) = S_n \times S_n \times \{I\}. \;\therefore\; \mathcal{ISOT}(G,\cdot) \succsim S_n \times S_n \times S_n.$$

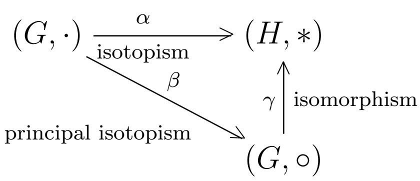
4 Smarandache f, g-Isotopes of Smarandache Loops

Theorem 4.1 *Let $(G,\cdot)$ and $(H,*)$ be S-groupoids. If $(G,\cdot)$ and $(H,*)$ are S-isotopic, then $(H,*)$ is S-isomorphic to some Smarandache principal isotope $(G,\circ)$ of $(G,\cdot)$.*

Proof
Since $(G,\cdot)$ and $(H,*)$ are S-isotopic S-groupoids with S-subsemigroups $(G_1,\cdot)$ and $(H_1,*)$, then there exist bijections $U,V,W : (G,\cdot) \to (H,*)$ such that the triple $\alpha = (U,V,W) : (G,\cdot) \to (H,*)$ is an isotopism and $\big(G_1\big)A = H_1 \ \forall\ A \in \{U,V,W\}$. To prove the claim of this theorem, it suffices to produce a closed binary operation '$*$' on G, bijections $X,Y : G \to G$, and bijection $Z : G \to H$ so that

- the triple $\beta = (X,Y,I) : (G,\cdot) \to (G,\circ)$ is a Smarandache principal isotopism and
- $Z : (G,\circ) \to (H,*)$ is an S-isomorphism or the triple $\gamma = (Z,Z,Z) : (G,\circ) \to (H,*)$ is an S-isotopism.

Thus, we need $(G,\circ)$ so that the commutative diagram below is true:



because following the proof of transitivity in Theorem 3.1, $\alpha = \beta\gamma$ which implies $(U,V,W) = (XZ,YZ,Z)$ and so we can make the choices; $Z = W$, $Y = VW^{-1}$, and $X = UW^{-1}$ and consequently,

$$x \cdot y = xUW^{-1} \circ VW^{-1} \iff x \circ y = xWU^{-1} \cdot yWV^{-1} \ \forall\ x,y \in G.$$

Hence, $(G,\circ)$ is a groupoid principal isotope of $(G,\cdot)$ and $(H,*)$ is an isomorph of $(G,\circ)$. It remains to show that these two relationships are Smarandache.

Note that $\big((H_1)Z^{-1},\circ\big) = (G_1,\circ)$ is a non-trivial subsemigroup in $(G,\circ)$. Thus, $(G,\circ)$ is an S-groupoid. So $(G,\circ) \succsim (H,*)$. $(G,\cdot)$ and $(G,\circ)$ are Smarandache principal isotopes because $(G_1)UW^{-1} = (H_1)W^{-1} = (H_1)Z^{-1} = G_1$ and $(G_1)VW^{-1} = (H_1)W^{-1} = (H_1)Z^{-1} = G_1$.

Corollary 4.1 *Let $(G,\cdot)$ be an S-groupoid with an arbitrary groupoid isotope $(H,*)$. Any such groupoid $(H,*)$ is an S-groupoid if and only if all the principal isotopes of $(G,\cdot)$ are S-groupoids.*

Proof
By classical result in principal isotopy [[11], III.1.4 Theorem], if $(G,\cdot)$ and $(H,*)$ are isotopic groupoids, then $(H,*)$ is isomorphic to some principal isotope $(G,\circ)$ of $(G,\cdot)$. Assuming $(H,*)$ is an S-groupoid then since $(H,*) \cong (G,\circ)$, $(G,\circ)$ is an S-groupoid. Conversely, let us assume all the principal isotopes of $(G,\cdot)$ are S-groupoids. Since $(H,*) \cong (G,\circ)$, then $(H,*)$ is an S-groupoid.

Theorem 4.2 *Let $(G,\cdot)$ be an S-quasigroup. If $(H,*)$ is an S-loop which is S-isotopic to $(G,\cdot)$, then there exist S-elements f and g so that $(H,*)$ is S-isomorphic to a Smarandache f,g principal isotope $(G,\circ)$ of $(G,\cdot)$.*

Proof
An S-quasigroup and an S-loop are S-groupoids. So by Theorem 4.1, $(H,*)$ is S-isomorphic to a Smarandache principal isotope $(G,\circ)$ of $(G,\cdot)$. Let $\alpha=(U,V,I)$ be the Smarandache principal isotopism of $(G,\cdot)$ onto $(G,\circ)$. Since $(H,*)$ is a S-loop and $(G,\circ)\succsim(H,*)$ implies that $(G,\circ)\cong(H,*)$, then $(G,\circ)$ is necessarily an S-loop and consequently, $(G,\circ)$ has a two-sided identity element say e and an S-subgroup $(G_2,\circ)$. Let $\alpha=(U,V,I)$ be the Smarandache principal isotopism of $(G,\cdot)$ onto $(G,\circ)$. Then,

$$xU\circ yV=x\cdot y\ \forall\ x,y\in G\Longleftrightarrow\ x\circ y=xU^{-1}\cdot yV^{-1}\ \forall\ x,y\in G.$$

So,

$$y=e\circ y=eU^{-1}\cdot yV^{-1}=yV^{-1}L_{eU^{-1}}\ \forall\ y\in G\ \text{and}\ x=x\circ e=xU^{-1}\cdot eV^{-1}=xU^{-1}R_{eV^{-1}}\ \forall\ x\in G.$$

Assign $f=eU^{-1}, g=eV^{-1}\in G_2$. This assignments are well defined and hence $V=L_f$ and $U=R_g$. So that $\alpha=(R_g,L_f,I)$ is a Smarandache f,g principal isotopism of $(G,\circ)$ onto $(G,\cdot)$. This completes the proof.

Corollary 4.2 *Let $(G,\cdot)$ be an S-quasigroup(S-loop) with an arbitrary groupoid isotope $(H,*)$. Any such groupoid $(H,*)$ is an S-quasigroup(S-loop) if and only if all the principal isotopes of $(G,\cdot)$ are S-quasigroups(S-loops).*

Proof
This follows immediately from Corollary 4.1 since an S-quasigroup and an S-loop are S-groupoids.

Corollary 4.3 *If $(G,\cdot)$ and $(H,*)$ are S-loops which are S-isotopic, then there exist S-elements f and g so that $(H,*)$ is S-isomorphic to a Smarandache f,g principal isotope $(G,\circ)$ of $(G,\cdot)$.*

Proof
An S-loop is an S-quasigroup. So the claim follows from Theorem 4.2.

5 G-Smarandache Loops

Lemma 5.1 *Let $(G,\cdot)$ and $(H,*)$ be S-isotopic S-loops. If $(G,\cdot)$ is a group, then $(G,\cdot)$ and $(H,*)$ are S-isomorphic groups.*

Proof
By Corollary 4.3, there exist S-elements f and g in $(G,\cdot)$ so that $(H,*) \succsim (G,\circ)$ such that $(G,\circ)$ is a Smarandache f,g principal isotope of $(G,\cdot)$.

Let us set the mapping $\psi := R_{f\cdot g} = R_{fg} : G \to G$. This mapping is bijective. Now, let us consider when $\psi := R_{fg} : (G,\cdot) \to (G,\circ)$. Since $(G,\cdot)$ is associative and $x\circ y = xR_g^{-1}\cdot yL_f^{-1}\ \forall\ x,y\in G$, the following arguments are true.

$x\psi\circ y\psi = x\psi R_g^{-1}\cdot y\psi L_f^{-1} = xR_{fg}R_g^{-1}\cdot yR_{fg}L_f^{-1} = x\cdot fg\cdot g^{-1}\cdot f^{-1}\cdot y\cdot fg = x\cdot y\cdot fg = (x\cdot y)R_{fg} = (x\cdot y)\psi\ \forall\ x,y\in G$. So, $(G,\cdot)\cong(G,\circ)$. Thus, $(G,\circ)$ is a group. If $(G_1,\cdot)$ and $(G_1,\circ)$ are the S-subgroups in $(G,\cdot)$ and $(G,\circ)$, then $\big((G_1,\cdot)\big)R_{fg} = (G_1,\circ)$. Hence, $(G,\cdot)\succsim(G,\circ)$.

$\therefore\ (G,\cdot)\succsim(H,*)$ and $(H,*)$ is a group.

Corollary 5.1 *Every group which is an S-loop is a GS-loop.*

Proof
This follows immediately from Lemma 5.1 and the fact that a group is a G-loop.

Corollary 5.2 *An S-loop is S-isomorphic to all its S-loop S-isotopes if and only if it is S-isomorphic to all its Smarandache f,g principal isotopes.*

Proof
Let $(G,\cdot)$ be an S-loop with arbitrary S-isotope $(H,*)$. Let us assume that $(G,\cdot)\succsim(H,*)$. From Corollary 4.3, for any arbitrary S-isotope $(H,*)$ of $(G,\cdot)$, there exists a Smarandache f,g principal isotope $(G,\circ)$ of $(G,\cdot)$ such that $(H,*)\succsim(G,\circ)$. So, $(G,\cdot)\succsim(G,\circ)$.

Conversely, let $(G,\cdot)\succsim(G,\circ)$, using the fact in Corollary 4.3 again, for any arbitrary S-isotope $(H,*)$ of $(G,\cdot)$, there exists a Smarandache f,g principal isotope $(G,\circ)$ of $(G,\cdot)$ such that $(G,\circ)\succsim(H,*)$. Therefore, $(G,\cdot)\succsim(H,*)$.

Corollary 5.3 *A S-loop is a GS-loop if and only if it is S-isomorphic to all its Smarandache f,g principal isotopes.*

Proof
This follows by the definition of a GS-loop and Corollary 5.2.

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