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Some Results in Fuzzy and Anti Fuzzy Group Theory

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Abstract: This paper is to further investigate some properties of an anti fuzzy subgroup of a group in relation to pseudo coset. It also uses isomorphism theorems to establish some results in relation to level subgroups of a fuzzy subgroup μ of a group G.

Key Words: Fuzzy group, level subgroup, Smarandache fuzzy algebra, anti fuzzy group, anti fuzzy subgroup, group homomorphism, group isomorphism.

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§1. Introduction

Major part of this work leans on the work of [5]. There are some new results using isomorphism theorems with some results in [5].

§2. Preliminaries

Definition 2.1 *Let X be a non-empty set. A fuzzy subset μ of the set G is a function $\mu : G \to [0, 1]$.*

Definition 2.2 *Let G be a group and μ a fuzzy subset of G. Then μ is called a fuzzy subgroup of G if*

(*i*) $\mu(xy) \geq \min\{\mu(x), \mu(y)\}$;
(*ii*) $\mu(x^{-1}) = \mu(x)$;
(*iii*)μ *is called a fuzzy normal subgroup if* $\mu(xy) = \mu(yx)$ *for all x and y in G.*

Definition 2.3 *Let G be a group and μ a fuzzy subset of G. Then μ is called an anti fuzzy subgroup of G if*

(*i*) $\mu(xy) \leq \max\{\mu(x), \mu(y)\}$;
(*ii*) $\mu(x^{-1}) = \mu(x)$.

Definition 2.4 *Let μ and λ be any two fuzzy subsets of a set X. Then*

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(i) λ *and* μ *are equal if* $\mu(x) = \lambda(x)$ *for every* x *in* X*;*
(ii) λ *and* μ *are disjoint if* $\mu(x) \neq \lambda(x)$ *for every* x *in* X*;*
(iii) $\lambda \subseteq \mu$ *if* $\mu(x) \geq \lambda(x)$*.*

Definition 2.5 *Let* μ *be a fuzzy subset (subgroup) of* X*. Then, for some* t *in* $[0, 1]$*, the set* $\mu_t = \{x \in X : \mu(x) \geq t\}$ *is called a level subset (subgroup) of the fuzzy subset (subgroup)* μ*.*

Remark 2.5.1 The set μ_t if it is group can be represented as G^t_μ.

Definition 2.6 *Let* μ *be a fuzzy subgroup of a group* G*. The set* $H = \{x \in G : \mu(x) = \mu(e)\}$ *is such that* $o(\mu) = o(H)$*.*

Definition 2.7 *Let* μ *be a fuzzy subgroup of a group* G*.* μ *is said to be normal if* $\sup \mu(x) = 1$ *for all* x *in* G*. It is said to be normalized if there is an* x *in* G *such that* $\mu(x) = 1$*.*

Definition 2.8 *Let* G *be a group and* μ *a fuzzy subset of* G*. Then* μ *is called an anti fuzzy subgroup of* G *if and only if* $\mu(xy^{-1}) \leq \max\{\mu(x), \mu(y)\}$*, and* μ *is called an anti fuzzy normal subgroup if* $\mu(xy) = \mu(yx)$ *for all* x *and* y*.*

Definition 2.9 *Let* μ *be a fuzzy subset of* X*. Then, for* $t \in [1, 0]$*, the set* $\mu_t = \{x \in X : \mu(x) \leq t\}$ *is called a lower level subset of the fuzzy subset* μ*.*

Definition 2.10 *Let* μ *be an anti fuzzy subgroup of* X*. Then, for* $t \in [1, 0]$*, the set* $\mu_t = \{x \in X : \mu(x) \leq t\}$ *is called a lower level subgroup of* μ*.*

Definition 2.11 *Let* μ *be an anti fuzzy subgroup of a group* G *of finite order. Then, the image of* μ *is* $Im(\mu) = \{t_i \in I : \mu(x) = t_i \text{ for some } x \text{ in } G\}$*, where* $I = [0, 1]$*.*

Definition 2.12 *Let* μ *be an anti fuzzy subgroup of a group* G*. For* a *in* G*, the anti fuzzy coset* $a\mu$ *of* G *determined by* a *and* μ *is defined by* $(a\mu)(x) = \mu(a^{-1}x)$ *for all* x *in* G*.*

Definition 2.13 *Let* μ *be an anti fuzzy subgroup of a group* G*. For* a *and* b *in* G*, the anti fuzzy middle coset* $a\mu b$ *of* G *is defined by* $(a\mu b)(x) = \mu(a^{-1}xb^{-1})$ *for all* x *in* G*.*

Definition 2.14 *Let* μ *be an anti fuzzy subgroup of* G *and an element* a *in* G*. Then pseudo anti fuzzy coset* $(a\mu)^p$ *is defined by* $(a\mu)^p(x) = p(a)\mu(x)$ *for all* x *in* G *and* p *in* P*.*

Definition 2.15 *The Cartesian product* $\lambda \times \mu : X \times Y \to [0, 1]$ *of two anti fuzzy subgroups is defined by* $(\lambda \times \mu)(x, y) = \max\{\lambda(x), \mu(y)\}$ *for all* (x, y) *in* $X \times Y$ *and* R_λ *is a binary anti fuzzy relation defined by* $R_\lambda(x, y) = \max\{\lambda(x), \lambda(y)\}$*. The anti fuzzy relation* R_λ *is said to be a similarity relation if*

(i) $R_\lambda(x, x) = 1$;
(ii) $R_\lambda(x, y) = R_\lambda(y, x)$;
(iii) $\max\{R_\lambda(x, y),\ R_\lambda(y, z)\} \leq R_\lambda(x, z)$.

Definition 2.16 *Let* G *be a finite group of order* n *and* μ *a fuzzy subgroup of* G*. Then for* t_1, t_2 *in* $[0, 1]$ *such that* $t_1 \leq t_2, \mu_{t_2} \subseteq \mu_{t_1}$*.*

Definition 2.17 *Let G be a finite group of order n and μ an anti fuzzy subgroup of G. Then for $t_1, t_2 \in [0,1]$ such that $t_1 \leq t_2, \mu_{t_1} \subseteq \mu_{t_2}$.*

Definition 2.18 *Let f be a group homomorphism from a group G to H. Then there is an isomorphism $\phi : f(G) \to G/Kerf$, where ϕ is the canonical isomorphism associated with f.*

Definition 2.19 *Let G be a group and H, K normal subgroups of G such that $H \leq K$. Then there is a natural isomorphism $G/K \cong (G/H)/(K/H)$.*

Proposition 2.20 *Let G be a group and μ a fuzzy subset of G. Then μ is a fuzzy subgroup of G if and only if G_μ^t is a level subgroup of G for every t in $[0, \mu(e)]$, where e is the identity of G.*

Proposition 2.21 *H as described in **2.6** can be realized as a level subgroup.*

Theorem 2.22 *G is a Dedekind or Hamiltonian group if and only if every fuzzy subgroup of G is fuzzy normal subgroup.* (A Dedekind and Hamiltonian groups have all the subgroups to be normal).

§3. Briefly on Properties of Anti Fuzzy Subgroup

Proposition 3.1 *Any two pseudo cosets of an anti fuzzy subgroup of a group G are either identical or disjoint.*

Proof Assume that $(a\mu)^p$ and $(b\mu)^p$ are any two identical pseudo anti fuzzy cosets of μ for any a and b in G. Then, $(a\mu)^p(x) = (b\mu)^p(x)$ for all x in G. Assume also on the contrary that they are disjoint. Then, there is no y in G such that $(a\mu)^p(y) = (b\mu)^p(y)$ which implies that $p(a)\mu(y) \neq p(b)\mu(y)$. The consequence is that $p(a) \neq p(b)$. This makes the assumption $(a\mu)^p(x) = (b\mu)^p(x)$ false.

Conversely, assume that $(a\mu)^p$ and $(b\mu)^p$ are disjoint, then $p(a)\mu(y) \neq p(b)\mu(y)$ for every y in G. But if it is assumed that this is also identical, then $p(a)\mu(y) = p(b)\mu(y)$ and that means $p(a) = p(b)$ so that $p(a)\mu(y) \neq p(b)\mu(y)$ cannot be true. □

Proposition 3.2 *Let μ be an anti fuzzy subgroup of any group G. Let $\{\mu_i\}$ be a partition of μ. Then*

(*i*) *each μ_i is normal if μ is normalized;*
(*ii*) *each μ_i is normal if μ is normal.*

Proof Note that for each i, $\mu_i \subseteq \mu$ which implies that $\mu_i(x) \leq \mu(x)$ for all x in G.

(*i*) Since μ is normalized, there is an x_0 in G such that $\mu_i(x) \leq \mu(x) \leq \mu(x_0) = 1$ for each i. Whence, $\mu_i(x) \leq 1$. Then $\sup \mu_i(x) = 1$.

(*ii*) Since μ is normal, $\sup \mu(x) = 1$, then $\mu(x) \leq 1$. Note that $\mu_i(x) \leq \mu(x) \leq 1$. Then $\mu_i(x) \leq 1$ and $\sup \mu_i(x) = 1$. □

Proposition 3.3 *Let μ be an anti fuzzy subgroup of any group G. Then $\mu(e) \leq 1$ even if μ is* *normalized.*

Proof Note that for all x in G, $0 \leq \mu(x) \leq 1$.

$$\mu(e) = \mu(xx^{-1}) \leq \max\{\mu(x), \mu(x^{-1})\} = \mu(x) \text{ since } \mu(x) = \mu(x^{-1}) \text{ for all } x \text{ in } G.$$

But since μ is normal, there is an x_0 in G such that $\mu(e) \leq \mu(x) \leq \mu(x_0) = 1$. Hence $\mu(e) \leq 1$. □

Proposition 3.4 *Let μ be an anti fuzzy subgroup of any group G and $R_\mu : G \times G \to [0,1]$ be given by $R_\mu(x,y) = \mu(xy^{-1})$. R_μ is not a similarity relation.*

Proof The reference [4] has shown that this is a similarity relation when μ is a fuzzy subgroup of G. But

$$R_\mu(x,x) = \mu(xx^{-1}) = \mu(e) \leq 1.$$

R_μ is not symmetric, hence not a similarity relation. □

§4. Application of Isomorphism Theorems of Groups to Fuzzy Subgroups

Proposition 4.1 *Let f be a group homomorphism between G and H. Let μ be a fuzzy subgroup of H. Then G is isomorphic to a level subgroup of H.*

Proof Since f is a homomorphism, it is defined on G.

$$Ker\ f = \{x \in G : f(x) = e_H\} \Leftrightarrow \{x \in G : \mu f(x) = \mu(e_H) \leq 1\}.$$

Hence, $\mu f(x) \leq 1$ for all x in G since μ is a fuzzy subgroup of H and $f(x)$ is in H.

$$Ker\ f = G \text{ so that } \mu f(G) \leq 1.$$

Also, note that

$$f(G) = \{y = f(x) \in H : \mu f(x) = \mu(y) = \mu(e_H)\}.$$

By 2.21 and 2.6, $f(G)$ is a level subgroup, say H_μ^t of H.

$$G/G = G \cong H_\mu^t$$

by Definition 2.18. □

Remark 4.2 It can be said then that every group G is isomorphic to a level subgroup of a group H if there is a group homomorphism between G and H and μ a fuzzy subgroup of H exits.

Proposition 4.3 *Let G be a Dedekind or an Hamiltonian group and μ a fuzzy subgroup of G. For $t_1, t_2 \in [0,1]$ such that $t_1 < t_2$ and $G/G_{1\mu}^t \cong (G/G_{2\mu}^t)/(G_{1\mu}^t/G_{2\mu}^t)$.*

Proof By Proposition 2.20, $G^t_{1\mu}$ and $G^t_{2\mu}$ are subgroups of G and by Theorem 2.22, they are normal subgroups. Also by Definition 2.16,

$$G^t_{2\mu} \leq G^t_{1\mu}.$$

Then, $f : G/G^t_{2\mu} \to G/G^t_{1\mu}$ is a group homomorphism and

$$Im(f) = G/G^t_{1\mu} \text{ if } f(gG^t_{2\mu}) = gG^t_{1\mu}.$$

Also, it can be shown that $Ker\ f = G^t_{1\mu}/G^t_{2\mu}$.

Then apply Definition 2.19 so that $G/G^t_{1\mu} \cong (G/G^t_{2\mu})/(t^t_{1\mu}/G^t_{2\mu})$. □

Remarks 4.4 It is equally of note that if μ is an anti fuzzy subgroup of a group G,

$$\text{for } t_1 < t_2,\ G^t_{1\mu} \leq G^t_{2\mu}$$

by Definition 2.17.

Following the same argument as in Proposition 4.3,

$$G/G^t_{2\mu} \cong (G/G^t_{1\mu})/(G^t_{2\mu}/G^t_{1\mu}).$$ □

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