ON SOME SERIES INVOLVING SMARANDACHE FUNCTION

by

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The study of infinite series involving Smarandache function is one of the most interesting aspects of analysis.

In this brief article me give only a bare introduction to it.

First we prove that the series $\sum_{K=2}^{\infty} \frac{S(k)}{(kH)!}$ converges and has the sum $\sigma \in \left]e-\frac{5}{2}, \frac{1}{2}\right[$.

S(m) is the Smarandache function: $S(m) = \min\{k \in \mathbf{N}; m | k!\}$.

Let us denote $1+\frac{1}{1!}+\frac{1}{2!}+\ldots+\frac{1}{n!}$ by E_n. We show that

$$E_{n+1}-\frac{5}{2} < \sum_{k=2}^{n} \frac{S(k)}{(k+1)!} < \frac{1}{2}$$ as follows:

$$\sum_{k=2}^{n} \frac{k}{(k+1)!} = \sum_{k=2}^{n}\left(\frac{1}{k!}-\frac{1}{(k+1)!}\right) = \sum_{k=2}^{n} \frac{1}{k!}-\sum_{k=2}^{n} \frac{1}{(k+1)!} = \frac{1}{2!}-\frac{1}{(n+1)!}$$

$S(k) \le k$ implies that $\sum_{k=2}^{n} \frac{S(k)}{(k+1)!} \le \sum_{k=2}^{n} \frac{k}{(k+1)!} = \frac{1}{2}-\frac{1}{(k+1)!} < \frac{1}{2}$.

On the other hand $k \geq 2$ implies that $S(k) > 1$ and consequently:

$$\sum_{k=2}^{n} \frac{S(k)}{(k+1)!} > \sum_{k=2}^{n} \frac{1}{(k+1)!} = \frac{1}{3!} + \frac{1}{4!} + \ldots + \frac{1}{n+1}! = E_{n+1} - \frac{5}{2} .$$

It follows that $E_{n+1} - \frac{5}{2} < \sum_{k=2}^{n} \frac{S(k)}{(k+1)!} < \frac{1}{2}$ and therefore $\sum_{k=2}^{\infty} \frac{S(k)}{(k+1)!}$ is a convergent series with sum $\sigma \in \left[e - \frac{5}{2}, \frac{1}{2} \right]$.

REMARK: Some of inequalities $S(k) \leq k$ are strictly and $k \geq S(k)+1$, $S(k) \geq 2$. Hence $\sigma \in \left] e - \frac{5}{2}, \frac{1}{2} \right[$.

We can also check that $\sum_{k=r}^{\infty} \frac{S(k)}{(k-r)!}$, $r \in \mathbb{N}^*$ and $\sum_{k=2}^{\infty} \frac{S(k)}{(k+r)!}$, $r \in \mathbb{N}$, are both convergent as follows:

$$\sum_{k=r}^{n} \frac{S(k)}{(k-r)!} \leq \sum_{k=r}^{n} \frac{k}{(k-r)!} = \frac{r}{0!} + \frac{r+1}{1!} + \frac{r+2}{2!} + \ldots + \frac{r+(n-r)}{(n-r)!} =$$

$$= r\left(\frac{1}{0!} + \frac{1}{1!} + \ldots + \frac{1}{(n-r)!}\right) + \left(\frac{1}{1!} + \frac{2}{2!} + \ldots + \frac{n-r}{(n-r)!}\right) = rE_{n-r} + E_{n-r-1}$$

We get $\sum_{k=r}^{n} \frac{S(k)}{(k-r)!} < rE_{n-r} + E_{n-r-1}$ which that $\sum_{k=r}^{\infty} \frac{S(k)}{(k-r)!}$ converges.

Also we have $\sum_{k=2}^{\infty} \frac{S(k)}{(k+r)!} < \infty$, $r \in \mathbb{N}$.

Let us define the set $M_2 = \left\{ m \in \mathbb{N} : m = \frac{n!}{2}, n \in \mathbb{N}, n \geq 3 \right\}$.

If $m \in M_2$ it is obvious that

$$S(m) = n, \quad m = \frac{n!}{2}. \; m \in M_2 \rightarrow \frac{m}{S(m)!} = \frac{\frac{n!}{2}}{n!} .$$

So, $\sum_{\substack{m=3 \\ m \in M_2}}^{\infty} \frac{m}{S(m)!} = \infty$ and therefore $\sum_{\substack{k=2 \\ k \in \mathbb{N}}}^{\infty} \frac{k}{S(k)!} = \infty$.

A problem: test the convergence behaviour of the series

$$\sum_{\substack{k=2 \\ k \in \mathbb{N}}}^{\infty} \frac{1}{S(k)!} .$$

REFERENCES

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