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Spherical Indicatrix Curves of Spatial Quaternionic Curves

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Abstract

In this paper, the spherical indicatrix curves drawn by quaternionic frenet vectors are computed. Also the quaternionic geodesic curvatures of the spherical indicatrix curves to E^3 and S^2 are found.

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1 Introduction

Quaternions were discovered, for the first time in 1843, by the Irish mathematician Sir William R. Hamilton [2]. Hamilton wanted to generalize complex numbers in order to be used in geometric optics. In 1987, Bharathi and Nagaraj defined the quaternionic curves in $\mathbb{E}^3$ and $\mathbb{E}^4$, they studied the differential geometry of space curves and introduced Frenet frames and formulae by using quaternions [4]. About a decade later, quaternionic inclined curves have been

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defined and harmonic curvatures studied by Karadağ and Sivridağ [6]. Tuna and Çöken have studied quaternion valued functions and quaternionic inclined curves in the semi-Euclidean space $\mathbb{E}_2^4$ [1]. They have given the Serret-Frenet formulae and they have defined quaternionic inclined curves and harmonic curvatures for the quaternionic curves in the semi-Euclidean space. Quaternionic rectifying curves have been studied by Güngör and Tosun, [5]. Şenyurt and Çalışkan have founded the Darboux vector of the spatial quaternionic curve according to the Frenet frame. Then, they calculated the curvature and torsion of the spatial quaternionic Smarandache curve formed by the unit Darboux vector with the normal vector, [7].

2 Preliminaries

In this section, we give the basic elements of the theory of quaternions and quaternionic curves. A more complete elementary treatment of quaternions and quaternionic curves can be found in [3] and [4], respectively. A real quaternion q is an expression of the form

$$q = d + ae_1 + be_2 + ce_3 \tag{2.1}$$

where $a, b, c \in \mathbb{R}$ and e_i, $1 \leq i \leq 3$, are quaternionic units which satisfy the non-commutative multiplication rules

$$\begin{cases} {e_1}^2 = {e_2}^2 = {e_3}^2 = e_1 \times e_2 \times e_3 = -1, \quad e_1, e_2, e_3 \in \mathbb{R}^3 \\ e_1 \times e_2 = e_3, e_2 \times e_3 = e_1, e_2 \times e_3 = e_1 \end{cases} \tag{2.2}$$

The algebra of the quaternions is denoted by Q and its natural basis is given by $\{e_1, e_2, e_3\}$. A general quaternion can be given by the form

$$q = S_q + V_q \tag{2.3}$$

where $S_q = d$ is the scalar part and $V_q = ae_1 + be_2 + ce_3$ is the vector part of q. The Hamilton conjugate of $q = S_q + V_q$ is defined by $\bar{q} = S_q - V_q$. Summation of two quaternions $q_1 = S_{q_1} + V_{q_1}$ and $q_2 = S_{q_2} + V_{q_2}$ is defined as $q_1 \oplus q_2 = (S_{q_1} + S_{q_2}) + (V_{q_1} + V_{q_2})$. Multiplication of a quaternion $q = S_q + V_q$ with a scalar $\lambda \in \mathbb{R}$ is identified as $\lambda \odot q = \lambda S_q + \lambda V_q$. If $\parallel q \parallel = 1$, then q is called unit quaternion. Let $q_1 = S_{q_1} + V_{q_1} = d_1 + a_1e_1 + b_1e_2 + c_1e_3$ and $q_2 = S_{q_2} + V_{q_2} = d_2 + a_2e_1 + b_2e_2 + c_2e_3$ be two quaternions in, then the quaternion product of q_1 and q_2 is given by

$$\begin{aligned} q_1 \times q_2 &= d_1d_2 - (a_1a_2 + b_1b_2 + c_1c_2) + (d_1a_2 + a_1d_2 + b_1c_2 - c_1b_2)e_1 \\ &+ (d_1b_2 + b_1d_2 + b_1a_2 - a_1b_2)e_2 + +(d_1c_2 + c_1d_2 + a_1b_2 - b_1a_2)e_3 \end{aligned}$$

or

$$q_1 \times q_2 = S_{q_1} S_{q_2} - \langle V_{q_1}, V_{q_2} \rangle + S_{q_1} V_{q_2} + S_{q_2} V_{q_1} + V_{q_1} \wedge V_{q_1} \tag{2.4}$$

where $\langle , \rangle$ and $\wedge$ denote the inner product and vector product in Euclidean 3-space.

In this space it is defined a symmetric real-valued, non-degenerate, bilinear form as follows:

$$\langle , \rangle_{|_Q} : Q \times Q \to \mathbb{R} \ , \langle q_1, q_2 \rangle|_Q = \frac{1}{2}(q_1 \times \bar{q}_2 + q_2 \times \bar{q}_1) \tag{2.5}$$

which is called the quaternion inner product. As a result the norm of q is

$$N(q) = \sqrt{q \times \bar{q}} = \sqrt{a^2 + b^2 + c^2 + d^2}. \tag{2.6}$$

The element q is called a spatial quaternion whenever $q + \bar{q} = 0$ and called a temporal quaternion whenever $q - \bar{q} = 0$. A general quaternion q can be given as $q = \frac{1}{2}(q + \bar{q}) + \frac{1}{2}(q - \bar{q})$. The three-dimensional Euclidean space is identified with the space of spatial quaternions [4].

The space $Q_H = \{q \in Q \mid q + \bar{q} = 0\}$ is defined in an obvious manner. Let $I = [0, 1]$ be an interval in the real line $\mathbb{R}$ and $s \in I$ be the arc-length parameter along the smooth curve

$$\gamma : [0, 1] \to Q_H, \ \ \gamma(s) = \sum_{i=1}^{3} \gamma_i(s) e_i. \tag{2.7}$$

The tangent vector $\gamma'(s) = t(s)$ has unit length $\|t(s)\| = 1$ for all s.

Let $\gamma : [0, 1] \to Q_H$ be a differentiable spatial quaternions curve with arc-length parameter s and $\{t(s), n_1(s), n_2(s)\}$ be the Frenet frame of γ at the point $\gamma(s)$, where

$$\begin{cases} t(s) = \gamma'(s) \\ n_1(s) = \dfrac{\gamma''(s)}{N(\gamma''(s))} \\ n_2(s) = t(s) \times n_1(s), \end{cases} \tag{2.8}$$

and if the curve $\gamma(s)$ is a non unit speed curve then we say that

$$\begin{cases} t(s) = \dfrac{\gamma'(s)}{\nu(s)}, \ \nu(s) = N(\gamma'(s)) \\ n_1(s) = n_2(s) \times t(s) \\ n_2(s) = \dfrac{\gamma'(s) \times \gamma''(s) + \nu(s)\nu'(s)}{N\big(\gamma'(s) \times \gamma''(s) + \nu(s)\nu'(s)\big)}. \end{cases} \tag{2.9}$$

Let $\{t(s), n_1(s), n_2(s)\}$ be the Frenet frame of $\gamma(s)$. Then Frenet formula, curvature and the torsion are given by

$$\begin{cases} t'(s) = k(s)n_1(s) \\ {n_1}'(s) = -k(s)t(s) + r(s)n_2(s) \\ {n_2}'(s) = -r(s)n_1(s), \end{cases} \tag{2.10}$$

and

$$\begin{cases} k(s) = \dfrac{N\big(\gamma'(s) \times \gamma''(s) + \nu(s)\nu'(s)\big)}{\nu(s)^3} \\ r(s) = \dfrac{\langle \gamma'(s) \times \gamma''(s), \gamma'''(s) \rangle_{|Q}}{\Big[N\big(\gamma'(s) \times \gamma''(s) + \nu(s)\nu'(s)\big)\Big]^2}, \end{cases} \tag{2.11}$$

where $t(s), n_1(s), n_2(s)$ are, respectively, the unit tangent, the unit principal normal and the unit binormal vector of a quaternionic curve [4]. The functions k, r are called the principal curvature and the torsion, respectively.

Considering the spatial quaternions curve $\gamma : [0,1] \to Q_H$, the moving frame $\{t(s), n_1(s), n_2(s)\}$ moves with a certain angular velocity around each axis for any $s \in [0,1]$. This axis is called instantaneous rotation axis of the spatial quaternionic curve. The Darboux axis vector in the direction indicated by D in defined as follows:

$$D = rt + kn_2. \tag{2.12}$$

Let D be the instantaneos Pfaff vector of curve γ. Let denote the angle between D and n_2 with φ,

$$\cos\varphi = \frac{k}{\sqrt{k^2 + r^2}}, \quad \sin\varphi = \frac{r}{\sqrt{k^2 + r^2}} \tag{2.13}$$

The unit vector of quaternionic Darboux vector [7] is indicated by w:

$$w = \frac{D}{N(D)} = \sin\varphi t + \cos\varphi n_2.$$

3 Spherical Indicatrix Curves of Spatial Quaternionic Curves

In this section we find the arc lenght of $(t), (n_1), (n_2)$ and (w) for the Spherical Indicatrix Curves, we compute the geodesic curvatures in E^3 and S^2.

We indicate the arc length for (t) with s_t, that is

$$
\begin{aligned}
s_t &= \int_0^s N(\frac{dt}{ds})ds \\
&= \int_0^s N(kn_1)ds \\
&= \int_0^s \sqrt{\frac{1}{2}(kn_1 \times \overline{kn_1}) + (kn_1 \times \overline{kn_1})}ds \\
&= \int_0^s kds. \qquad (3.1)
\end{aligned}
$$

If the arc length for $(n_1), (n_2)$ and (w) are s_{n_1}, s_{n_2} and s_w, we have:

$$
\begin{aligned}
s_{n_1} &= \int_0^s N(\frac{dn_1}{ds})ds \\
&= \int_0^s N(-kt + rn_2)ds \\
&= \frac{1}{\sqrt{2}} \int_0^s \sqrt{((-kt + rn_2) \times (\overline{-kt + rn_2}) + (-kt + rn_2) \times (\overline{-kt + rn_2}))}ds \\
&= \frac{1}{\sqrt{2}} \int_0^s \sqrt{2(k^2 + r^2)}ds \\
&= \int_0^s N(w(s))ds, \qquad (3.2)
\end{aligned}
$$

$$
\begin{aligned}
s_{n_2} &= \int_0^s N(\frac{dn_2}{ds})ds \\
&= \int_0^s N(-rn_2)ds \\
&= \frac{1}{\sqrt{2}}\int_0^s \sqrt{(-rn_2 \times \overline{-rn_2}) - (rn_2 \times \overline{-rn_2})}ds \\
&= \int_0^s rds, \qquad (3.3)
\end{aligned}
$$

$$
\begin{aligned}
s_w &= \int_0^s N(\frac{dw}{ds})ds \\
&= \int_0^s N\Big(\frac{d(\sin\varphi t + \cos\varphi n_2)}{ds}\Big)ds \\
&= \int_0^s N(\varphi'(\cos\varphi t - \sin\varphi n_2))ds \\
&= \int_0^s \varphi' ds. \qquad (3.4)
\end{aligned}
$$

Let $\alpha_t(s_t) = t(s)$ be a unit speed regular spherical curves. We denote s_t as the arc-lenght parameter of tangents indicatrix (t)

$$\alpha_t(s_t) = t(s) \qquad (3.5)$$

Let us consider the tangent vector t_t of curve (t). Supposing that the tangent vector of the (t) geodeseic curvature is Λ_t, we get

$$\Lambda_t = N(D_{t_t}t_t). \qquad (3.6)$$

Differentiating (3.5), we have

$$\frac{d\alpha_t}{ds_t}\frac{ds_t}{ds} = t'(s)$$

and

$$t_t \frac{ds_t}{ds} = kn_1. \tag{3.7}$$

From the equation (3.7)

$$t_t(s_t) = n_1(s).$$

Computing the derivatives and after some algebra we get,

$$D_{t_t} t_t = -t + \frac{r}{k} n_2. \tag{3.8}$$

By substituting (3.5), we obtain

$$\begin{aligned}
\Lambda_t &= \sqrt{\frac{1}{2}(-t + \frac{r}{k} n_2) \times \overline{(-t + \frac{r}{k} n_2)}}, \\
\Lambda_t &= \sqrt{1 + (\frac{r}{k})^2}
\end{aligned}$$

or from the equation (2.13),

$$\Lambda_t = \sec \varphi. \tag{3.9}$$

Likewise, let us suppose that the tangent vector of the curve (n_1) is t_{n_1}. Supposing that Λ_{n_1} is the geodesic curvature of (n_1) at E^3, we get

$$\Lambda_{n_1} = \sqrt{1 + \left(\frac{\varphi'}{N(w)}\right)^2}. \tag{3.10}$$

Let us define the tangent vector of the curve (n_2) as t_{n_2}. Supposing that Λ_{n_2} is the geodesic curvature of (n_2) at E^3, we get

$$\begin{aligned}
\Lambda_{n_2} &= \sqrt{1 + (\frac{k}{r})^2} \\
\Lambda_{n_2} &= \csc \varphi. \qquad (3.11)
\end{aligned}$$

Following a similar approach, we consider the tangent vector is t_w of the curve (w). Supposing that Λ_w is the geodesic curvature of (w) at E^3, we get

$$\Lambda_w = \sqrt{1 + \left(\frac{N(w)}{\varphi'}\right)^2}. \tag{3.12}$$

Let find the geodesic curvatures of $(t), (n_1), (n_2)$ and (w) to S^2. Supposing that the geodesic curvature for (t) is Γ_t, it is $\Gamma_t = N(\overline{D}_{t_t} t_t)$. Because of the Gauss equation and $S(t_t) = t_t$, we have

$$\begin{aligned} \overline{D}_{t_t} t_t &= D_{t_t} t_t + \langle S(t_t), t_t \rangle_{|Q} t, \\ \overline{D}_{t_t} t_t &= D_{t_t} t_t + t, \\ \overline{D}_{t_t} t_t &= \frac{r}{k} n_2 = \tan \phi n_2. \end{aligned}$$

Then,

$$\Gamma_t = \tan \phi \tag{3.13}$$

will be achieved. Likewise, Supposing that the geodesic curvature for (n_1) is Γ_{n_1}, it is $\Gamma_{n_1} = N(\overline{D}_{t_{n_1}} t_{n_1})$. Because $\langle S(t_{n_1}), t_{n_1} \rangle_{|Q} = 1$ at the statement

$$\begin{aligned} \overline{D}_{t_{n_1}} t_{n_1} &= D_{t_{n_1}} t_{n_1} + \langle S(t_{n_1}), t_{n_1} \rangle_{|Q} t, \\ \overline{D}_{t_{n_1}} t_{n_1} &= \frac{\varphi'}{N(w)} (\sin \varphi t + \cos \varphi n_2) \end{aligned}$$

will be found out. Then,

$$\Gamma_{n_1} = \frac{\varphi'}{N(w)} \tag{3.14}$$

will be achieved. Likewise, supposing that the geodesic curvatures for (n_2) are (w) are Γ_{n_2} and Γ_w, these are $\Gamma_{n_2} = N(\overline{D}_{t_{n_2}} t_{n_2})$ and $\Gamma_w = N(\overline{D}_{t_w} t_w)$. Then,

$$\begin{aligned} \Gamma_{n_2} &= \cot \varphi, \\ \Gamma_w &= \frac{N(w)}{\varphi'}. \end{aligned}$$

will be achieved.

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