An Integer as a Sum of Consecutive Integers

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Abstract: This is a simple study of expressions of positive integers as sums of consecutive integers. In the first part proof is given for the fact that N can be expressed in exactly d(L)-1 ways as a sum of consecutive integers, L is the largest odd factor of N and d(L) is the number of divisors of L. In the second part answer is given to the question: Which is the smallest integer that can be expressed as a sum of consecutive integers in n ways.

Introduction: There is a remarkable similarity between the four definitions given below. The first is the well known Smarandache Function. The second function was defined by K. Kashihara and was elaborated on in his book *Comments and Topics on Smarandache Notions and Problems*[1]. This function and the Smarandache Ceil Function were also examined in the author's book *Surfing on the Ocean of Numbers*[2]. These three functions have in common that they aim to answer the question **which is the smallest positive integer N which possesses a certain property pertaining to a given integer n.** It is possible to pose a large number of questions of this nature.

1. **The Smarandache Function S(n):**
   S(n)=N where N is the smallest positive integer which divides n!.

2. **The Pseudo-Smarandache Function Z(n):**
   Z(n)=N where N is the smallest positive integer such that 1+2+...+N is divisible by n.

3. **The Smarandache Ceil Function of order k, $S_k(n)$:**
   $S_k(n)=N$ where N is he smallest positive integer for which n divides N^k

4. **The n-way consecutive integer representation R(n):**
   R(n)=N where N is the smallest positive integer which can be represented as a sum of consecutive integer is n ways.

There may be many positive integers which can be represented as a sum of positive integers in n distinct ways - but which is the smallest of them? This article gives the answer to this question. In the study of R(n) it is found that the arithmetic function d(n), the number of divisors of n, plays an important role.

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[1] Erhus University Press, 1996
[2] Erhus University Press, 1997

Question 1: In how many ways n can a given positive integer N be expressed as the sum of consecutive positive integers?

Let the first term in a sequence of consecutive integers be Q and the number terms in the sequence be M. We have N=Q+(Q+1)+ ... +(Q+M-1) where M>1.

$$N = \frac{M(2Q+M-1)}{2} \tag{1}$$

or

$$Q = \frac{N}{M} - \frac{M-1}{2} \tag{2}$$

For a given positive integer N the number of sequences n is equal to the number of positive integer solutions to (2) in respect of Q. Let us write $N=L\cdot 2^s$ and $M=m\cdot 2^k$ where L and m are odd integers. Furthermore we express L as a product of any of its factors $L=m_1 m_2$. We will now consider the following cases:

1. $s=0, k=0$
2. $s=0, k\neq 0$
3. $s\neq 0, k=0$
4. $s\neq 0, k\neq 0$

Case 1. s=0, k=0.

Equation (2) takes the form

$$Q = \frac{m_1 m_2}{m} - \frac{m-1}{2} \tag{3}$$

Obviously we must have $m\neq 1$ and $m\neq L$ (=N).

For $m=m_1$ we have $Q>0$ when $m_2-(m_1-1)/2>0$ or $m_1<2m_2+1$. Since m_1 and m_2 are odd, the latter inequality is equivalent to $m_1<2m_2$ or, since $m_2=N/m_1$, $m_1 < \sqrt{2N}$.

We conclude that a factor m ($\neq 1$ and $\neq N$) of N (odd) for which $m < \sqrt{2N}$ gives a solution for Q when M=m is inserted in equation (2).

Case 2. s=0, k≠0.

Since N is odd we see form (2) that we must have k=1. With M=2m equation (2) takes the form

$$Q = \frac{m_1 m_2}{2m} - \frac{2m-1}{2} \tag{4}$$

For m=1 (M=2) we find Q=(N-1)2 which corresponds to the obvious solution $\frac{N-1}{2}+\frac{N+1}{2}=N$.

Since we can have no solution for m=N we now consider $m=m_2$ ($\neq 1$, $\neq N$). We find $Q=(m_1-2m_2+1)/2$. Q>0 when $m_1>2m_2-1$ or, since m_1 and m_2 are odd, $m_1>2m_2$ Since $m_1m_2=N$, $m_2=N/m_1$ we find $m>\sqrt{2N}$.

We conclude that a factor m ($\neq 1$ and $\neq N$) of N (odd) for which $m>\sqrt{2N}$ gives a solution for Q when M=2m is inserted in equation (2).

The number of divisors of N is known as the function d(N). Since all factors of N except 1 and N provide solutions to (2) while M=2, which is not a factor of N, also provides a solution (2) we find that the number of solutions n to (2) when N is odd is

$$n=d(N)-1 \qquad (5)$$

Case 3. s≠0, k=0.

Equation (2) takes the form

$$Q=\frac{1}{2}\left(\frac{m_1m_2}{m}2^{s+1}-m+1\right) \qquad (6)$$

Q≥1 requires $m^2<L\cdot 2^{s+1}$. We distinguish three cases:

Case 3.1. k=0, m=1. There is no solution.

Case 3.2. k=0, $m=m_1$. Q≥1 for $m_1<m_2 2^{s+1}$ with a solution for Q when $M=m_1$.

Case 3.3. k=0, $m=m_1m_2$. Q≥1 for $L<2^{s+1}$ with a solution for Q when M=L.

Case 4. s≠0, k≠0.

Equation (2) takes the form

$$Q=\frac{1}{2}\left(\frac{m_1m_2}{m}2^{s-k+1}-m\cdot 2^k+1\right) \qquad (7)$$

Q is an integer if and only if m divides L and $2^{s-k+1}=1$. The latter gives k=s+1. Q≥1 gives

$$Q=\frac{1}{2}\left(\frac{m_1m_2}{m}+1\right)-m\cdot 2^s\geq 1 \qquad (8)$$

Again we distinguish three cases:

Case 4.1. $k=s+1$, $m=1$. $Q\geq 1$ for $L>2^{s+1}$ with a solution for Q when $M=2^{s+1}$

Case 4.2. $k=s+1$, $m=m_2$ $Q\geq 1$ for $m_1>m_2 2^{s+1}$ with a solution for Q when $M=m_2 2^{s+1}$

Case 4.3. $k=s+1$, $m=L$ $Q\geq 1$ for $1-L\cdot 2^s\geq 1$. No solution

Since all factors of L except 1 provide solutions to (2) we find that the number of solutions n to (2) when N is even is

$$n=d(L)-1 \tag{9}$$

Conclusions:

- The number of sequences of consecutive positive integers by which a positive integer $N=L\cdot 2^s$ where $L\equiv 1 \pmod 2$ can be represented is $n=d(L)-1$.
- We see that the number of integer sequences is the same for $N=2^sL$ and $N=L$ no matter how large we make s.
- When $L<2^s$ the values of M which produce integer values of Q are odd, i.e. N can in this case only be represented by sequences of consecutive integers with an odd number of terms.
- There are solutions for all positive integers L except for L=1, which means that $N=2^s$ are the only positive integers which cannot be expressed as the sum of consecutive integers.
- $N=P\cdot 2^s$ has only one representation which has a different number of terms (<p) for different s until $2^{s+1}>P$ when the number of terms will be p and remain constant for all larger s.

A few examples are given in table 1.

Table 1. The number of sequences for L=105 is 7 and is independent of s in $N=L\cdot 2^s$.

N=105 s=0		N=210 s=1		N=3360 s=5 $L>2^{s+1}$		N=6720 s=6 $L<2^{s+1}$	
Q	M	Q	M	Q	M	Q	M
34	3	69	3	1119	3	2239	3
19	5	40	5	670	5	1342	5
12	7	27	7	477	7	957	7
1	14	7	15	217	15	441	15
6	10	1	20	150	21	310	21
15	6	12	12	79	35	175	35
52	2	51	4	21	64	12	105

Question 2: Which is the smallest positive integer N which can be represented as a sum of consecutive positive integers in n different ways.

We can now construct the smallest positive integer R(n)=N which can be represented in n ways as the sum of consecutive integers. As we have already seen this smallest integer is necessarily odd and satisfies n=d(N)-1.

With the representation $N = p_1^{\alpha_1} p_2^{\alpha_2} ... p_j^{\alpha_j}$ we have

$$d(N)=(\alpha_1+1)(\alpha_2+1)...(\alpha_j+1)$$

or

$$n+1=(\alpha_1+1)(\alpha_2+1)...(\alpha_j+1) \qquad (10)$$

The first step is therefore to factorize n+1 and arrange the factors (α_1+1), (α_2+1) ... (α_j+1) in descending order. Let us first assume that $\alpha_1>\alpha_2> ... >\alpha_j$ then, remembering that N must be odd, the smallest N with the largest number of divisors is

$$R(n)= N = 3^{\alpha_1} 5^{\alpha_2} 7^{\alpha_3} ... p_j^{\alpha_j}$$

where the primes are assigned to the exponents in ascending order starting with $p_1=3$. Every factor in (10) corresponds to a different prime even if there are factors which are equal.

Example: $n = 269$
$n+1= 2\cdot3^3\cdot5 = 5 \cdot3 \cdot3 \cdot3 \cdot2$
$R(n)= 3^4\cdot5^2\cdot7^2\cdot11^2\cdot13=156080925$

When n is even it is seen from (10) that α_1, α_2, ... α_j must all be even. In other words the smallest positive integer which can be represented as a sum of consecutive integers in a given number of ways must be a square. It is therefore not surprising that even values of n in general generate larger smallest N than odd values of n. For example, the smallest integer that can be represented as a sum of integers in 100 ways is $N=3^{100}$, which is a 48-digit integer, while the smallest integer that can be expressed as a sum of integer in 99 ways is only a 7-digit integer, namely 3898125.

Conclusions:

- 3 is always a factor of the smallest integer that can be represented as a sum of consecutive integers in n ways.
- The smallest positive integer which can be represented as a sum of consecutive integers in given even number of ways must be a square.

Table 2. The smallest integer R(n) which can be represented in n ways as a sum of consecutive positive integers.

n	R(n)	R(n) in factor form
1	3	3
2	9	3^2
3	15	$3 \cdot 5$
4	81	3^4
5	45	$3^2 \cdot 5$
6	729	3^6
7	105	$3 \cdot 5 \cdot 7$
8	225	$3^2 5^2$
9	405	$3^4 5$
10	59049	3^{10}
11	315	$3^2 5 \cdot 7$
12	531441	3^{12}