Available online at www.sciencedirect.com
ScienceDirect
Chaos, Solitons and Fractals 33 (2007) 315–334
CHAOS
SOLITONS & FRACTALS
www.elsevier.com/locate/chaos

The generalized golden proportions, a new theory of real numbers, and ternary mirror-symmetrical arithmetic

Alexey Stakhov

The International Club of the Golden Section, 6 McCreary Trail, Bolton, ON, Canada L7E 2C8

Accepted 4 January 2006

Communicated by Professor G. Iovane

Abstract

We consider two important generalizations of the golden proportion: golden p-proportions [Stakhov AP. Introduction into algorithmic measurement theory. Soviet Radio, Moscow, 1977 [in Russian]] and "metallic means" [Spinadel VW. La familia de números metálicos en Diseño. Primer Seminario Nacional de Gráfica Digital, Sesión de Morfología y Matemática, FADU, UBA, 11–13 Junio de 1997, vol. II, ISBN 950-25-0424-9 [in Spanish]; Spinadel VW. New smarandache sequences. In: Proceedings of the first international conference on smarandache type notions in number theory, 21–24 August 1997. Lupton: American Research Press; 1997, p. 81–116. ISBN 1-879585-58-8]. We develop a constructive approach to the theory of real numbers that is based on the number systems with irrational radices (Bergman's number system and Stakhov's codes of the golden p-proportions). It follows from this approach ternary mirror-symmetrical arithmetic that is the basis of new computer projects.

1. Introduction

A theory of Fibonacci numbers and the golden section is an important branch of modern mathematics [1–3]. Let us consider the basic notions of this theory. It is known from the ancient times that a problem of the "*division of a line segment AB with a point C in the extreme and middle ratio*" reduces to the following:

$$\frac{AB}{CB} = \frac{CB}{AC}. \tag{1}$$

This problem came to us from the "Euclidean Elements". In modern science the problem is known as the *golden section problem*. Its solution reduces to the following algebraic equation:

$$x^2 = x + 1. \tag{2}$$

E-mail address: goldenmuseum@rogers.com
URL: www.goldenmuseum.com

0960-0779/$ - see front matter © 2006 Elsevier Ltd. All rights reserved.
doi:10.1016/j.chaos.2006.01.028

Its positive root

$$\tau = \frac{1+\sqrt{5}}{2} \tag{3}$$

is called the *golden proportion*, the *golden mean*, or the *golden ratio*.

It follows from (2) the following remarkable identity for the golden ratio:

$$\tau^n = \tau^{n-1} + \tau^{n-2} = \tau \times \tau^{n-1}, \tag{4}$$

where n takes its values from the set: $0, \pm 1, \pm 2, \pm 3, \ldots$

The *Fibonacci numbers*

$$F_n = \{1, 1, 2, 3, 5, 8, 13, 21, 34, 55, \ldots\} \tag{5}$$

are a numerical sequence given by the following recurrence relation:

$$F_n = F_{n-1} + F_{n-2} \tag{6}$$

with the initial terms

$$F_1 = F_2 = 1. \tag{7}$$

The *Lucas numbers*

$$L_n = \{1, 3, 4, 7, 11, 18, 29, 47, 76, \ldots\} \tag{8}$$

are a numerical sequence given by the following recurrence relation:

$$L_n = L_{n-1} + L_{n-2} \tag{9}$$

with the initial terms

$$L_1 = 1; \quad L_2 = 3. \tag{10}$$

The Fibonacci and Lucas numbers can be extended to the negative values of the index n. The "*extended*" *Fibonacci and Lucas numbers* are presented in Table 1.

As can be seen from Table 1, the terms of the "extended" sequences F_n and L_n have a number of wonderful mathematical properties. For example, for the odd $n = 2k + 1$ the terms of the sequences F_n and F_{-n} coincide, that is, $F_{2k+1} = F_{-2k-1}$, and for the even $n = 2k$ they are equal and of opposite sign, that is, $F_{2k} = -F_{-2k}$. As to the Lucas numbers L_n, here all are opposite, that is, $L_{2k} = L_{-2k}$; $L_{2k+1} = -L_{-2k-1}$.

In the last decades this theory and its applications were developing intensively. In this connection the works on the generalization of Fibonacci numbers and golden proportion are of the greatest interest. There are two of the most important generalizations of the golden proportion. The first of them, which was called the *golden p-proportions* ($p = 0, 1, 2, 3, \ldots$), was made by the present author in a book [4] published in 1977. The author's works [4–21] in this field develops and generalizes the theory of Fibonacci numbers and the golden proportion and gives many interesting applications of the golden p-proportions to the different fields of mathematics and computer science.

Another generalization of the golden proportion called the *metallic means* or *metallic proportions* was developed by the Argentinean mathematician Vera Spinadel in the series of the papers and books [22–32]. The first Spinadel's paper in this area [22] was published in 1997.

The first purpose of the present paper is to discuss two important generalizations of the golden proportions stated in the works [4–32]. The second purpose is to show how these generalizations, in particular, a concept of the golden p-proportions [4] can influence the development of number theory. The third purpose is to put forward an original computer arithmetic project namely the *ternary mirror-symmetrical arithmetic* developed in [12].

Table 1
The "extended" Fibonacci and Lucas numbers

n	0	1	2	3	4	5	6	7	8	9	10
F_n	0	1	1	2	3	5	8	13	21	34	55
F_{-n}	0	1	−1	2	−3	5	−8	13	−21	34	−55
L_n	2	1	3	4	7	11	18	29	47	76	123
L_{-n}	2	−1	3	−4	7	−11	18	−29	47	−76	123

2. Generalizations of the golden proportions

2.1. The golden p-proportions

In the author's book [4] the following geometric problem called a *problem of the golden p-section* was formulated. Let us consider a given number $p = 0, 1, 2, 3, \ldots$ and divide a line segment AB with a point C in the following proportion:

$$\frac{CB}{AC} = \left(\frac{AB}{CB}\right)^p. \tag{11}$$

As is shown in [4], a solution of the problem (11) reduces to the following algebraic equation:

$$x^{p+1} = x^p + 1, \tag{12}$$

where $p = 0, 1, 2, 3, \ldots$

A positive root τ_p of Eq. (12) was named in [4] the *golden p-proportion* because for the case $p = 1$ the root τ_p coincides with the classical golden proportion (3). Eq. (12) was named in [16] the "*golden*" *algebraic equation* because for the case $p = 1$ Eq. (12) reduces to Eq. (2).

Note that for the case $p = 0$ the number $\tau_p = \tau_0 = 2$ and for the case $p = \infty$, $\tau_p = \tau_\infty = 1$. It means that on the numerical axis between the numerical constants 2 and 1 there are an infinite number of the numerical constants τ_p that are positive roots of the "golden" algebraic equation (12).

It follows directly from (12) the following identity that connects the golden p-proportions degrees:

$$\tau_p^n = \tau_p^{n-1} + \tau_p^{n-p-1} = \tau_p \times \tau_p^{n-1}. \tag{13}$$

Note that for the case $p = 1$ the identity (13) reduces to identity (4).

The golden p-proportions τ_p can be got from the combinatorial consideration [4]. Let us consider Pascal triangle (Table 2).

Note that the binomial coefficient C_n^k is on the intersection of the nth column ($n = 0, 1, 2, 3, \ldots$) and the kth row ($k = 0, 1, 2, 3, \ldots$) of Pascal triangle.

It is well known that the following identity that connects the binomial coefficients with the binary numbers:

$$2^n = C_n^0 + C_n^1 + C_n^2 + \cdots + C_n^n. \tag{14}$$

Let us shift now every row of Pascal triangle by one column to the right with respect to previous row and consider the following "*deformed*" *Pascal triangle* (Table 3).

Table 2
Pascal triangle

1	1	1	1	1	1	1	1	1	1
	1	2	3	4	5	6	7	8	9
		1	3	6	10	15	21	28	36
			1	4	10	20	35	56	84
				1	5	15	35	70	126
					1	6	21	56	126
						1	7	28	84
							1	8	36
								1	9
									1
1	2	4	8	16	32	64	128	256	512

Table 3
The "deformed" Pascal triangle

1	1	1	1	1	1	1	1	1	1	1	1
		1	2	3	4	5	6	7	8	9	10
				1	3	6	10	15	21	28	36
						1	4	10	20	35	56
								1	5	15	35
										1	6
1	1	2	3	5	8	13	21	34	55	89	144

If we sum binomial coefficients of the "deformed" Pascal triangle by columns we get unexpectedly *Fibonacci numbers* 1, 1, 2, 3, 5, 8, 13,... Here the sum of the binomial coefficients of the nth row is equal to the $(n+1)$th Fibonacci number F_{n+1} ($F_{n+1} = F_n + F_{n-1}$; $F_1 = F_2 = 1$).

It is easy to prove that the Fibonacci number F_{n+1} can be represented via binomial coefficients as follows:

$$F_1(n+1) = C_n^0 + C_{n-1}^1 + C_{n-2}^2 + C_{n-3}^3 + c_{n-4}^4 + \cdots \tag{15}$$

If we shift every row of the initial Pascal triangle in p columns to the right with respect to the previous row ($p = 0, 1, 2, 3, \ldots$), then we get a table of binomial coefficients (a new "deformed" Pascal triangle). If we sum binomial coefficients of a new "deformed" Pascal triangle called *Pascal p-triangle* we will get the so-called *generalized Fibonacci numbers or Fibonacci p-numbers* [4] that are given with the following recurrence relation:

$$F_p(n+1) = F_p(n) + F_p(n-p) \quad \text{for } n > p; \tag{16}$$

$$F_p(1) = F_p(2) = \cdots = F_p(p+1) = 1. \tag{17}$$

It was proved in [4] that the ratio of the adjacent Fibonacci p-numbers $F_p(n)/F_p(n-1)$ for the case $n \to \infty$ tends to the golden p-proportion τ_p. It means that the numbers τ_p express some deep mathematical properties of Pascal triangle.

It was proved in [4] that the Fibonacci p-number $F_p(n+1)$ can be represented via binomial coefficients as follows:

$$F_p(n+1) = C_n^0 + C_{n-p}^1 + C_{n-2p}^2 + C_{n-3p}^3 + C_{n-4p}^4 + \cdots \tag{18}$$

It is clear that the Fibonacci p-numbers given with (16)–(18) are a wide generalization of the "binary" numbers 1, 2, 4, 8, 16, ... ($p = 0$) and Fibonacci numbers 1, 1, 2, 3, 5, 8, 13, ... ($p = 1$). Note that the formulas (14) and (15) are partial cases of the formula (18) for the cases $p = 0$ and $p = 1$, respectively.

The notions of the golden p-proportions and Fibonacci p-numbers became a source of the original mathematical concepts, in particular, "Golden" Algebraic Equations [17], Generalized Principle of the Golden Section [20], Generalized Binet Formulas and Lucas p-numbers [18], Harmony Mathematics [9,21], Algorithmic Measurement Theory [4,5,7], Codes of the Golden Proportion [6,14] and so on.

2.2. The metallic means family of Vera Spinadel

In 1997, the Argentinean mathematician Vera Spinadel considered a very interesting generalization of Fibonacci numbers and the golden proportion [22–32]. Let us consider the following recurrence formula:

$$G(n+1) = pG(n) + qG(n-1), \tag{19}$$

where p and q are natural numbers. From (19) we get

$$\frac{G(n+1)}{G(n)} = p + q\frac{G(n-1)}{G(n)} = p + \frac{q}{\frac{G(n)}{G(n-1)}}. \tag{20}$$

Taking the limits in both members and assuming that $\lim_{n\to\infty} \frac{G(n+1)}{G(n)}$ exists and is equal to a real number x, we have

$$x = p + \frac{q}{x} \tag{21}$$

or

$$x^2 - px - q = 0. \tag{22}$$

The algebraic equation (22) has a positive solution

$$\sigma_p^q = \frac{p + \sqrt{p^2 + 4q}}{2}. \tag{23}$$

This means that

$$\lim_{n\to\infty} \frac{G(n+1)}{G(n)} = \sigma_p^q. \tag{24}$$

Formula (23) forms a *metallic means family* (MMF) of Vera Spinadel. All the members of the MMF are positive quadratic irrational numbers σ_p^q that are the positive solutions of quadratic equations of the type (22).

Let us begin with $x^2 - px - 1 = 0$. Then it is very easy to find the members of the MMF that satisfy this equation, expanding them in continued fractions. In fact, if $p = q = 1$, formula (21) can be written as follows:

$$\sigma_1^1 = 1 + \frac{1}{x}. \tag{25}$$

Replacing iteratively the value of x of the second term, we have

$$\sigma_1^1 = 1 + \cfrac{1}{1 + \cfrac{1}{1 + \cdots}},$$

that is,

$$\sigma_1^1 = [1, 1, 1, \ldots] = [\bar{1}]. \tag{26}$$

It is clear that expression (26) defines the golden proportion in the form of a purely periodic continued fraction, that is,

$$\sigma_1^1 = \tau = \frac{1+\sqrt{5}}{2} = [\bar{1}].$$

Analogously, if $p = 2$ and $q = 1$, we obtain the *silver mean*

$$\sigma_2^1 = 1 + \sqrt{2} = 2 + \cfrac{1}{2 + \cfrac{1}{2 + \cfrac{1}{2 + \cfrac{1}{2 + \ldots}}}} = [\bar{2}] \tag{27}$$

another purely periodic continued fraction.

If $p = 3$ and $q = 1$, we get the *bronze mean*

$$\sigma_3^1 = \frac{3+\sqrt{13}}{2} = [\bar{3}].$$

For $p = 4$; $q = 1$, the metallic mean $\sigma_4^1 = 2 + \sqrt{5} = [\bar{4}] = \tau^3$, a striking result related to the continued fraction expansion of odd powers of the golden proportion.

It is easy to verify that the following metallic means are

$$\sigma_5^1 = \frac{5+\sqrt{29}}{2} = [\bar{5}]; \quad \sigma_6^1 = 3 + \sqrt{10} = [\bar{6}]; \quad \sigma_7^1 = \frac{7+\sqrt{53}}{2} = [\bar{7}]; \quad \sigma_8^1 = 4 + \sqrt{17} = [\bar{8}]; \quad \sigma_9^1 = \frac{9+\sqrt{85}}{2}$$
$$= [\bar{9}]; \quad \sigma_{10}^1 = 5 + \sqrt{26} = [\overline{10}].$$

If, instead, we consider equation $x^2 - x - q = 0$, we have for $q = 1$, again the golden proportion. If $p = 1$ and $q = 2$, we obtain the *copper mean* $\sigma_1^2 = 2 = [2, \bar{0}]$, a periodic continued fraction. If $p = 1$ and $q = 3$, we get the *nickel mean*, $\sigma_1^3 = \lim_{n\to\infty} \frac{G(n+1)}{G(n)} = \frac{1+\sqrt{13}}{2} = [2, \bar{3}]$. Analogously

$$\sigma_1^4 = [2, \overline{1, 1, 3}]; \quad \sigma_1^5 = [2, \overline{1, 3}]; \quad \sigma_1^6 = 3 = [3, \bar{0}]; \quad \sigma_1^7 = [3, \bar{5}]; \quad \sigma_1^8 = [3, \overline{2, 1, 2, 5}];$$
$$\sigma_1^9 = [3, \overline{1, 1, 5}]; \quad \sigma_1^{10} = [3, \overline{1, 2, 2, 1, 5}]; \quad \sigma_1^{11} = [3, \overline{1, 5}]; \quad \sigma_1^{12} = 4 = [4, \bar{0}]$$

and all these members of the MMF are of the form $[m, \overline{n_1, n_2, \ldots}]$.

Vera Spinadel shows in her works [22–32] a number of interesting applications of the metallic means: Pisot and Salem numbers, quasi-crystals and so on.

3. Bergman's number system

In 1957, the American mathematician George Bergman introduced the positional number system of the following kind [33]:

$$A = \sum_i a_i \tau^i, \tag{28}$$

where A is some real number and a_i is the ith digit binary numeral, 0 or 1, $i = 0, \pm1, \pm2, \pm3$, τ^i is the weight of the ith digit, $\tau = \frac{1+\sqrt{5}}{2}$ is the base or radix of the number system (3).

Remind that the classical "binary" number system used widely in modern computers has the following form:

$$A = \sum_i a_i 2^i, \tag{29}$$

where A is some real number and a_i is the ith digit binary numeral, 0 or 1, $i = 0, \pm1, \pm2, \pm3$, 2^i is a weight of the ith digit, 2 is the base or radix of the number system (20).

At first glance, there does not exist any difference between formulas (28) and (29) but it is only at the first glance. A principal distinction between these number systems consists in the fact that Bergman used the irrational number $\tau = \frac{1+\sqrt{5}}{2}$ (the golden proportion) as the base of the number system (28). That is why Bergman called it a *number system with an irrational base or "Tau System"*. Although Bergman's paper [33] contained a result of great importance for number system theory, however, in that period it simply was not noted either by mathematicians or engineers. In the conclusion of his paper [33] Bergman wrote: "*I do not know of any useful application for systems such as this, except as a mental exercise and pastime, though it may be of some service in algebraic number theory*".

Let us consider representations of numbers in the "Tau System" (28). It is clear that the abridged representation of number A given with (28) has the following form:

$$A = a_n a_{n-1} \dots a_1 a_0, a_{-1} a_{-2} \dots a_{-m}. \tag{30}$$

A representation (30) is called a *"golden" representation of A*.

We can see that the "golden" representation of A is some binary code combination, which is separated with a comma into two parts, the left-hand part $a_n a_{n-1} \dots a_1 a_0$ corresponding to the "weights" $\tau^n, \tau^{n-1}, \dots, \tau^1, \tau^0 = 1$ and the right-hand part $a_{-1} a_{-2} \dots a_{-m}$ corresponding to the "weights" with negative powers: $\tau^{-1}, \tau^{-2}, \dots, \tau^{-m}$. Note that the "weights" τ^i $(i = 0, \pm 1, \pm 2, \pm 3, \dots)$ are given with the mathematical formula (4).

The "Tau System" (28) overturns our traditional ideas about number systems and has the following unusual properties [33]:

(1) One may represent in (28) some irrational numbers (the powers of the golden ratio τ^i and their sums) by using finite numbers of bits that it is impossible in classical number systems (decimal, binary etc.). For example, the "golden" representation 100101 is the abridged notation of the irrational number $8 + 3\sqrt{5}$. The radix τ of the "Tau System" is represented in traditional manner, that is

$$\tau = \frac{1+\sqrt{5}}{2} = 10.$$

(2) The "golden" representations of natural numbers have a form of fractional numbers and consist of a finite number of bits. Table 4 gives the "golden" representations of some natural numbers in (28).

(3) Each non-zero number have many "golden" representations in (28). Here the different "golden" representations of the same real number A can be got one from another by using the special transformations, "*convolution*" and "*devolution*", carried out on the initial "golden" representation. These special code transformations are based on the identity (4). The "devolution" is defined as the following transformation carried out on the three neighboring digits of the initial "golden" representation

$$100 \to 011. \tag{31}$$

For example, we can perform the following "devolutions" over the "golden" representation of the number:

$$5 = 1000,\ 1001 = 0110,\ 0111 = 0101,\ 1111 \text{ (maximal form)}. \tag{32}$$

The "convolution" is called a back transformation, that is

$$011 \to 100. \tag{33}$$

Table 4
The "golden" representations of natural numbers

N	$\sum_i a_i \tau^i$	Abridged notation
1	τ^0	1, 0
2	$\tau^1 + \tau^{-2}$	10, 01
3	$\tau^2 + \tau^{-2}$	100, 01
4	$\tau^2 + \tau^0 + \tau^{-2}$	101, 01
5	$\tau^3 + \tau^{-1} + \tau^{-4}$	1000, 1001
6	$\tau^3 + \tau^1 + \tau^{-4}$	1010, 0001
7	$\tau^4 + \tau^{-4}$	10000, 0001
8	$\tau^4 + \tau^0 + \tau^{-4}$	10001, 0001
9	$\tau^4 + \tau^1 + \tau^{-2} + \tau^{-4}$	10010, 0101
10	$\tau^4 + \tau^2 + \tau^{-2} + \tau^{-4}$	10100, 0101

For example, we can perform the following "convolutions" over the "golden" representation 0101, 1111 taken from example (32):

$$5 = 0101,\ 1111 = 0110,\ 0111 = 1000,\ 1001 \quad \text{(minimal form)}. \tag{34}$$

The notions of "convolution" and "devolution" form the basis of two special "golden" representations called a "*minimal form*" and a "*maximal form*" [6]. If we perform all possible "convolutions" in the "golden" representation of number A we will come to the "minimal form" of A (see the example (34)); if we perform all possible "devolutions" in the "golden" representations of A we will come to the "maximal" form (see the example (33)). Note that the "minimal" and "maximal" forms are the extreme "golden" representations possessing the following specific properties namely, (a) *in the "minimal" there are no contiguous 1's*; (b) *in the "maximal" form there are no contiguous 0's*.

4. Codes of the golden proportion

Let us consider an infinite set of the "standard segments" based on the golden p-proportion τ_p

$$G_p = \{\tau_p^n\}, \tag{35}$$

where $n = 0, \pm 1, \pm 2, \pm 3 \ldots;$ τ_p^n are the golden p-proportion powers connected among themselves with the identity (13).

The set (35) "generates" the following constructive method of the real number A representation called the *code of the golden p-proportion* [6]

$$A = \sum_i a_i \tau_p^i, \tag{36}$$

where $a_i \in \{0, 1\}$ and $i = 0, \pm 1, \pm 2, \pm 3, \ldots$

Let us consider the partial cases of the number representation (36). It is clear that for the case $p = 0$ the formula (36) reduces to (29). For the case $p = 1$ the formula (36) reduces to Bergman's number system (28). For other cases of p the formula (36) "generates" an infinite number of new positional representations called *codes of the golden p-proportions*.

5. A new approach to geometric definition of real numbers

5.1. Euclidean definition of natural numbers

A number is one of the most important mathematical concepts. During many millennia this concept is widened and made more precise. A discovery of irrational numbers led to the concept of real numbers that include natural numbers, rational and irrational numbers. Further development of the number concept is connected with introduction of complex numbers and their generalizations, the hypercomplex numbers. Let us start from the concept of natural numbers concept that underlies the basis of number theory. We know from the "Euclidean Elements" the following "geometric approach" to the natural number definition.

Let

$$S = \{1, 1, 1, \ldots\} \tag{37}$$

be an infinite set of geometric segments called "*Monads*" or "*Units*". Then according to Euclid we can define a natural number N as a sum of "Monads"

$$N = 1 + 1 + 1 + \ldots + 1 \ (N \text{ times}). \tag{38}$$

Despite the limiting simplicity of such definition, it has played a large role in mathematics and underlies many useful mathematical concepts, for example, concepts of *prime* and *composite* numbers, and also concept of *divisibility*, one of the main concepts of number theory.

5.2. A constructive approach to the number concept

It is known as the so-called "constructive approach" to the number definition of real numbers based on the "Dichotomy Principle". According to the "constructive approach" any *constructive real number A* is some mathematical object given by the mathematical formula (29).

A number definition (29) has the following geometric interpretation. Let

$$B = \{2^n\} \tag{39}$$

be the set of the geometric line segments of the length 2^n ($n = 0, \pm 1, \pm 2, \pm 3, \ldots$). Then all the geometric line segments that can be represented as the final sum (29) can be defined as *constructive real numbers*.

It is clear that the definition (29) selects from the set of all real numbers only some part of real numbers that can be represented in the form (29). All the rest real numbers that cannot be represented in the form of the sum (29) are called *non-constructive*. It is clear that all irrational numbers, in particular, the mathematical constants π, e, the golden ratio refer to the non-constructive numbers. But in the framework of the definition (29) we have to refer to the non-constructive numbers some rational numbers (for example, 2/3, 3/7 and so on) that are known as *periodical fractions* that cannot be represented in the form of the final sum (29).

Note that although the number definition (29) restricts considerably the set of real numbers this fact does not diminish its importance from the "practical", computational point of view. It is easy to prove that any non-constructive real number can be represented in the form (29) approximately; here the approximation error Δ would be decreased with no limit if we increase a number of terms in (29). However, $\Delta \neq 0$ for all non-constructive real numbers. By essence, we use in modern computers only the constructive real numbers given by (29).

5.3. Newton's definition of real numbers

Within many millennia the mathematicians developed and made more precise a concept of a number. In the 17th century during origin of modern science and mathematics the new methods of study of the "continuous" processes (integration and differentiation) are developed and a concept of real numbers again goes out on the foreground. Most clearly a new definition of this concept was given by Isaac Newton, one of the mathematical analysis founders, in his "Arithmetica Universalis" (1707):

> "*We understand under number not the set of units but the abstract ratio of any magnitude to other one of the same kind taken by us as the unit*".

This definition gives us a uniform definition of all real numbers, rational and irrational. If we consider now the "Euclidean definition" (38) from the point of view of "Newton's definition" then the "Monad" 1 plays the role of the Unit in it. In the "binary" notation (29) the number 2, the radix of the number system, plays the role of the Unit.

5.4. A new constructive definition of real number

From Newton's point of view we can consider the codes of the golden p-proportion given by (36) as a new definition of real numbers.

We proved above that the "binary" number system (29) and Bergman's number system (28) are partial cases of the codes of the golden p-proportion (36). Finally, let us consider the case $p \to \infty$. For this case it is possible to show, that $\tau_p \to 1$; it means that the positional representation (26) reduces in the limit to the Euclidean definition (38).

Note that for the case $p > 0$ the radix τ_p of the positional number system (36) is an irrational number. It means that we have come to the *number systems with irrational radices* that are principally new class of the positional number systems. Note that for the case $p = 1$ the number system (36) reduces to Bergman's number system (28) that was introduced by the American mathematician George Bergman in 1957 [33]. Note that Bergman's number system (28) is the first number system with irrational radix in the history of mathematics.

It follows from the given consideration that the positional representation (36) is a very wide generalization of the classical binary number system (29), Bergman's number system (28) and the Euclidean definition (38) that are partial cases of the general representation (36).

6. Some properties of the number systems with irrational radices

6.1. "Constructive" and "non-constructive" real numbers

Note that expression (36) divides a set of real numbers into two non-overlapping subsets, the "constructive" real numbers, which can be represented in the form of the final sum (36), and all the remaining real numbers, which cannot be represented in the form of the sum (36) (the "non-constructive" real numbers). Such approach to real numbers is distinguished radically from the classical approach when a set of real numbers is divided into rational and irrational numbers.

Really, all powers of the golden p-proportion of the kind τ_p^i $(i = 0, \pm 1, \pm 2, \pm 3, \ldots)$, which are irrational numbers, can be represented in the form (36), that is, they refer to the subset of the "constructive" numbers. For example,

$$\begin{aligned} \tau_p^1 &= 10 \quad \tau_p^{-1} = 0.1, \\ \tau_p^2 &= 100 \quad \tau_p^{-2} = 0.01, \\ \tau_p^3 &= 1000 \quad \tau_p^{-3} = 0.001. \end{aligned}$$

It follows from the definition (36) that all real numbers, which are the sum of the golden p-proportion powers, are "constructive" numbers of the kind (36). For example, according to (36) the real number $A = \tau_p^2 + \tau_p^{-1}\tau_p^{-3}$ can be represented in the following abridged form:

$$A = 100,\ 101.$$

Note that a possibility of representation of some irrational numbers (the powers of the golden p-proportion and their sums) in the form of the final totality of bits is the first unusual property of the number systems (36).

6.2. Representation of natural numbers

Let us consider the representation of natural numbers in the form (36)

$$N = \sum_i a_i \tau_p^i, \tag{40}$$

where N is some natural number, τ_p is the radix of number system (40), $a_i \in \{0, 1\}$, $i = 0, \pm 1, \pm 2, \pm 3, \ldots$

It is proved in [6] that all the representations of the kind (40) are *finite*, that is, for a given $p \geqslant 0$ any sum (40) consists of a finite number of terms. For example, for the case $p = 1$ (Bergman's number system) the initial terms of natural series can be represented as in Table 1.

Let us formulate this result as the following theorem.

Theorem 1. *All natural numbers are "constructive" real numbers, that is, for a given* $p = 0, 1, 2, 3, \ldots$ *any natural number can be represented as a finite sum* (40).

7. The Z-, D- and Z_p-properties of natural numbers

Let us consider now the representation of the arbitrary natural number of N in Bergman's number system (28)

$$N = \sum_i a_i \tau^i. \tag{41}$$

The representation (41) is called the τ-code of natural number N.

Let us apply Binet's formula [1–3]

$$\tau^i = \frac{L_i + F_i\sqrt{5}}{2} \tag{42}$$

to the τ-code (41) and represent the formula (41) in the following form:

$$2N = \sum_i a_i L_i + \sqrt{5}\sum_i a_i F_i. \tag{43}$$

We can see that the left-hand part of the expression (43) is a natural number, which is even always. The right-hand part of (43) is the sum of two terms. The former is the sum of the Lucas numbers with the binary coefficients. Because every Lucas number is an integer (see Table 1) then any sum of Lucas numbers is integer.

Let us consider now the latter term. This one is the product of the irrational number $\sqrt{5}$ by the sum of Fibonacci numbers with the binary coefficients 0 or 1. Because every Fibonacci number is an integer (see Table 1) then any sum of Fibonacci numbers is an integer always. Thus, identity (43) asserts that the natural (even) number $2N$ is equal to the sum of some integer (the sum of Lucas number with the binary coefficients) and the product of other integer (the sum of Fibonacci numbers with the binary coefficients) by the irrational number $\sqrt{5}$.

There arises a question: for what conditions identity (43) is true for arbitrary natural number N? The answer is unique. This is possible for the following condition: if the sum of Fibonacci numbers in the identity (43) equals to 0 identically, that is

$$\sum_i a_i F_i = 0. \tag{44}$$

On the other hand, the number $2N$ is even. If we take into consideration identity (44) then the sum of Lucas numbers in identity (44) has to be even always.

Let us analyze the sum of $\sum_i a_i L_i$ and $\sum_i a_i F_i$ in identity (43). These sums were obtained as a result of substitution of Binet's formula (42) into expression (41), which gives the representation of natural number N in Bergman's number system (28). It means that all the binary numerals a_i in the sums of $\sum_i a_i L_i$ and $\sum_i a_i F_i$ coincide with the corresponding coefficients a_i, in the τ-code (41).

Thus, as a result of this consideration, we came to the following mathematical discovery in number theory that we can formulate as the following theorems.

Theorem 2 (Z-property of natural numbers). *If we represent any natural number N in the τ-code (41) and then replace in it all powers of the golden ratio τ^i ($i = 0, \pm1, \pm2, \pm3, \ldots$) with the corresponding Fibonacci numbers F_i, then the sum obtained in this manner has to be equal to 0 identically independently from the initial natural number N.*

Theorem 3 (D-property of natural numbers). *If we represent any natural number N in the τ-code (41) and then replace in it all powers of the golden ratio τ^i ($i = 0, \pm1, \pm2, \pm3, \ldots$) with the corresponding Lucas numbers L_i, then the sum obtained in this manner has to be an even number, which equals to the double value of the initial natural number N.*

In [14] it is proved the following theorem that sets the Z_p-property of natural numbers.

Theorem 4 (Z_p-property of natural numbers). *If we represent any natural number N in the code of the golden p-proportion (36) where $p = 1, 2, 3, \ldots$, and then replace in it all powers of the golden p-proportion τ_p^i ($p = 1, 2, 3, \ldots$; $i = 0, \pm1, \pm2, \pm3, \ldots$) with the corresponding Fibonacci p-numbers $F_p(i)$, then the sum obtained in this manner has to be equal to 0 identically independently from the initial natural number N.*

Of what practical importance could be these new fundamental properties of natural numbers, Z-property and D-property? To answer this question let us imagine some hypothetical computer network, which uses the τ-code (41) or code of the golden p-proportion (40) for number representation. For this hypothetical case the new properties of natural numbers formulated above can be considered as a *new universal check indication of code information in computer networks*.

8. The *F*- and *L*-codes

We can get two new representations of natural number N if we use the τ-code (41) and the Z-property (44). Taking into consideration the Z-property (44), expression (43) can be written in the following form:

$$2N = \sum_i a_i L_i + \sum_i a_i F_i \quad \text{or} \quad N = \sum_i a_i \frac{L_i + F_i}{2} = \sum_i a_i F_{i+1} \tag{45}$$

as $F_{i+1} = \frac{L_i + F_i}{2}$ [1–3].

The expression (45) is called the F-code of natural number N [14].

Note that the corresponding binary numerals in the τ-code (41) and in the F-code (45) coincide. This means that the F-code (45) can be got from the τ-code (41) of the same natural number N by using the simple replacing of the golden ratio power τ^i in the formula of (41) with the Fibonacci number F_{i+1}, where $i = 0, \pm1, \pm2, \pm3, \ldots$

Let us represent now the F-code (45) in the following form:

$$N = \sum_i a_i F_{i+1} + 2\sum_i a_i F_i = \sum_i a_i L_{i+1} \tag{46}$$

as $L_{i+1} = F_{i+1} + 2F_i$ [1–3].

The expression (16) is called the *L-code* of natural number N [14].

Note that the binary numerals in the representations (41) and (46) coincide. It follows from this that the L-code (46) can be got from the τ-code (41) of the same natural number N by simple replacing of the golden ratio power τ_i in the τ-representation (41) with the Lucas number L_{i+1}, where $i = 0, \pm1, \pm2, \pm3, \ldots$

It is clear that the L-code (46) can be got also from the F-code (45) of the same number N by simple replacing of the Fibonacci number F_{i+1} in the F-code (45) of the same natural number N with the Lucas number L_{i+1}.

Thus, it follows from the above consideration that there are three different representations of one and the same natural number N: τ-code (41), F-code (45) and L-code (46).

As an example, let us consider the code representation of the decimal number 10 in the τ-code (41) (see Table 4)

$$10 = 1\,0\,1\,0\,0,\ 0\,1\,0\,1. \tag{47}$$

It is clear that the code representation (47) has the following algebraic interpretation:

$$10 = \tau^4 + \tau^2 + \tau^{-2} + \tau^{-4}. \tag{48}$$

Using Binet's formula (42) we can represent the sum (48) as follows:

$$10 = \tau^4 + \tau^2 + \tau^{-2} + \tau^{-4} = \frac{L_4 + F_4\sqrt{5}}{2} + \frac{L_2 + F_2\sqrt{5}}{2} + \frac{L_{-2} + F_{-2}\sqrt{5}}{2} + \frac{L_{-4} + F_{-4}\sqrt{5}}{2}. \tag{49}$$

If we take into consideration the following correlations connecting the "extended" Fibonacci and Lucas numbers (see Table 1):

$$L_{-2} = L_2; \quad L_{-4} = L_4; \quad F_{-2} = -F_2; \quad F_{-4} = -F_4$$

we will get the following result from the expression (49):

$$10 = \frac{2(L_4 + L_2)}{2} = L_4 + L_2 = 7 + 3.$$

Let us consider the algebraic interpretation of the code combination (47) as the F- and L-codes

$$10 = F_5 + F_3 + F_{-1} + F_{-3} = 5 + 2 + 1 + 2;$$
$$10 = L_5 + L_3 + L_{-1} + L_{-3} = 11 + 4 - 1 - 4.$$

9. The ternary mirror-symmetrical representation

9.1. Conversion of the binary golden representation to the ternary golden representation

Let us consider the τ-code (41) of natural number N for the case of its "minimal form". It means that each binary unit $a_k = 1$ in the τ-code (41) would be "enclosed" with the two next binary zeros $a_{k-1} = a_{k+1} = 0$.

Let us consider now the following expression for the powers of the golden ratio:

$$\tau^k = \tau^{k+1} - \tau^{k-1}. \tag{50}$$

Expression (50) has the following code interpretation [12]:

$$\begin{array}{ccc} k+1 & k & k-1 \\ 0 & 1 & 0 \\ \uparrow & \downarrow & \uparrow \end{array} = \begin{array}{ccc} k-1 & k & k-1 \\ 1 & 0 & \bar{1} \\ & & \end{array}, \tag{51}$$

where $\bar{1}$ is a negative unit, i.e. $\bar{1} = -1$. It follows from (51) that the positive binary 1 of the kth digit is transformed into two 1's, the positive 1 of the $(k+1)$th digit and the negative $\bar{1}$ of the $(k-1)$th digit.

The code transformation (51) might be used for the conversion of the "minimal form" of τ-code (41) into the *ternary τ-representation*.

Let us convert the "minimal form" of the τ-code of number

$$5 = \begin{array}{ccccccccc} 4 & 3 & 2 & 1 & 0 & -1 & -2 & -3 & -4 \\ 0 & 1 & 0 & 0 & 0, & 1 & 0 & 0 & 1 \\ \uparrow & \downarrow & \uparrow & & \uparrow & \downarrow & \uparrow & & \end{array} \tag{52}$$

into the ternary τ-representation". To do this we have to apply the code transformation (51) simultaneously to all the digits, which are the binary 1's and have odd indices ($k = 2m + 1$). We can see that the transformation (51) could be applied for the situation (52) only to the 3rd and (-1)th digits, which are the binary 1's. As the result of such transformation of (52) we will get the following ternary representation of number 5:

$$5 = \begin{array}{ccccccccc} 4 & 3 & 2 & 1 & 0 & -1 & -2 & -3 & -4 \\ 1 & 0 & \bar{1} & 0 & 1, & 0 & \bar{1} & 0 & 1 \end{array} \tag{53}$$

We can see from (53) that the digits with the even indices are equal to 0 identically but the digits with the odd indices take the ternary values from the set $\{\bar{1}, 0, 1\}$. This means that all the digits with the even indices are "non-informative" because their values are equal to 0 identically. Omitting in (53) all the "non-informative" digits we will get the following ternary representation of the initial number N:

$$N = \sum_i b_{2i}\tau^{2i}, \tag{54}$$

where b_{2i} is the ternary numeral of the $(2i)$th digit.

Let us introduce the following digit enumeration for the ternary representation (54). Every ternary digit b_{2i} is replaced with the ternary digit c_i. As a result of such enumeration we will get expression (54) in the following form:

$$N = \sum_i c_i\tau^{2i}, \tag{55}$$

where c_i is the ith digit ternary numeral; τ^{2i} is a weight of the ith digit; τ^2 is the radix of the number system (55). We will name representation (55) a *ternary τ-representation or ternary τ-system*.

With regard to the expression (55) the ternary τ-representation (52) of the number 5 takes the following form:

$$5 = \overset{2}{1}\,\overset{1}{\bar{1}}\,\overset{0}{1},\,\overset{-1}{\bar{1}}\,\overset{-2}{1} \tag{56}$$

The conversion of the τ-code (41) to the ternary τ-representation (55) could be performed [11] by means of a simple combinative logical circuit, which transforms the next three binary digits $a_{2i+1}a_{2i}a_{2i-1}$ of the initial τ-code (51) represented in the "minimal form" into the ternary informative digit $b_{2i} = c_i$, of the ternary τ-representation in accordance with Table 5.

Note that Table 5 uses only 5 binary code combinations from 8 because the initial "binary" τ-code (41) is represented in the "minimal form" and the code combinations 0 1 1, 1 1 0, 1 1 1 are forbidden for it.

The code transformations given with the 2nd and 4th rows of Table 5 are trivial. The code transformations given with the 3rd, 5th and 6th rows of Table 5 follow from (51). For instance, the code transformation of the 6th row 1 0 1 ⇒ 0 means that the negative $1(\bar{1})$ arising in accordance with (52) from the left-hand binary digit $a_{2i+1} = 1$ is summarized with the positive 1 arising from the right-hand binary digit $a_{2i-1} = 1$. It follows from here that their sum equals to $c_i = 0$.

Let us consider now the "ternary F- and L-representations" following from the ternary τ-code (41). It is clear that the expressions for the ternary F- and L-representations have the following forms, respectively:

$$N = \sum_i c_i F_{2i+1}, \tag{57}$$

$$N = \sum_i c_i L_{2i+1}. \tag{58}$$

Note that the values of the ternary digits in the representations (55), (57) and (58) coincide. It follows from this consideration that the ternary τ-, F-, and L-representations of the number 5 from example (56) have the following algebraic interpretations:

(a) The ternary τ-representation,

$$5 = 1 \times \tau^4 + \bar{1} \times \tau^2 + 1 \times \tau^0 + \bar{1} \times \tau^{-2} + 1 \times \tau^{-4}$$
$$= \frac{L_4 + F_4\sqrt{5}}{2} - \frac{L_2 + F_2\sqrt{5}}{2} + 1 - \frac{L_{-2} + F_{-2}\sqrt{5}}{2} + \frac{L_{-4} + F_{-4}\sqrt{5}}{2} = L_4 - L_2 + 1 = 7 - 3 + 1.$$

(b) The ternary F-representation,

$$5 = 1 \times F_5 + \bar{1} \times F_3 + 1 \times F_1 + \bar{1} \times F_{-1} + 1 \times F_{-3} = 5 - 2 + 1 - 1 + 2.$$

Table 5
A transformation of the "binary" τ-code into the ternary informative digit

a_{2i+1}	a_{2i}	a_{2i-1}	c_i
0	0	0	0
0	0	1	1
0	1	0	1
1	0	0	$\bar{1}$
1	0	1	0

(c) The ternary L-representation,

$$5 = 1 \times L_5 + \bar{1} \times L_3 + 1 \times L_1 + \bar{1} \times L_{-1} + 1 \times L_{-3} = 11 - 4 + 1 + 1 - 4.$$

9.2. Representation of negative numbers

An important advantage of the number system (55) is a possibility to represent both positive and negative numbers in "direct" code. A ternary representation of the negative number $(-N)$ can be got from the ternary τ-representation of the initial number N by means of application of the rule of a "ternary inversion"

$$1 \to \bar{1}, \quad 0 \to 0, \quad \bar{1} \to 1. \tag{59}$$

Applying this rule to the ternary τ-representation (56) of the number 5 we can get the ternary τ-representation of the negative number (-5)

$$-5 = \begin{array}{ccccc} 2 & 1 & 0 & -1 & -2 \\ 1 & \bar{1} & 1, & \bar{1} & 1 \end{array}$$

9.3. Mirror-symmetrical property of integer representation

Considering the ternary τ-representation (56) of the number 5 we can see that the left-hand part $(1\ \bar{1})$ of the ternary τ-representation (56) is mirror-symmetrical to its right-hand part $(\bar{1}\ 1)$ relatively to the 0th digit. It is proved in [12] that this property of the "mirror symmetry" is a fundamental property of integer representations, which arises at their representation in the "Ternary Tau System" (55). Table 6 demonstrates this property for some natural numbers.

Thus, thanks to this simple investigation we have discovered one more fundamental property of integers, the property of "*mirror symmetry*", which appears at their representation in the "Ternary Tau System" (45). That is why the "ternary τ-system" (58) is called the *Ternary Mirror-Symmetrical Number System* [12].

9.4. The radix of the ternary τ-system

It follows from (55) that the radix of the ternary τ-system (55) is a square of the golden proportion:

$$\tau^2 = \frac{3 + \sqrt{5}}{2} \approx 2.618$$

It means that the number system (55) is a number system with an irrational radix.

The radix of the number system (55) has the following traditional representation:

$$\tau^2 = 10.$$

Table 6
Mirror-symmetrical property of integers

i	3	2	1	0	-1	-2	-3
τ^{2i}	τ^6	τ^4	τ^2	τ^0	τ^{-2}	τ^{-4}	τ^{-6}
F_{2i+1}	13	5	2	1	1	2	5
L_{2i+1}	29	11	4	1	-1	-4	-11
N							
0	0	0	0	0,	0	0	0
1	0	0	0	1,	0	0	0
2	0	0	1	$\bar{1}$,	1	0	0
3	0	0	1	0,	1	0	0
4	0	0	1	1,	1	0	0
5	0	1	$\bar{1}$	1,	$\bar{1}$	1	0
6	0	1	0	$\bar{1}$,	0	1	0
7	0	1	0	0,	0	1	0
8	0	1	0	1,	0	1	0
9	0	1	1	$\bar{1}$,	1	1	0
10	0	1	1	0,	1	1	0

9.5. A range of number representation

Let us consider the range of number representation in the ternary τ-system (55). Suppose that the ternary τ-representation (55) has $2m+1$ ternary digits. In this case one may represent in the number system (55) all integers in the range from

$$N_{\max} = \underbrace{11\ldots1}_{m}\ 1,\ \underbrace{11\ldots1}_{m} \tag{60}$$

to

$$N_{\min} = \underbrace{\bar{1}\bar{1}\ldots\bar{1}}_{m}\ \bar{1},\ \underbrace{\bar{1}\bar{1}\ldots\bar{1}}_{m}. \tag{61}$$

It is clear that $N_{\min}$ is the ternary inversion of $N_{\max}$, i.e.

$$|N_{\min}| = N_{\max}.$$

It follows from this consideration that by means of the $(2m+1)$-digits we can represent in the ternary mirror-symmetrical number system (55):

$$2N_{\max}+1 \tag{62}$$

integers including the number 0.

For calculation of $N_{\max}$ let us use the ternary L-representation (58). Then we can interpret the code combination (50) as the abridged notation of the following sum:

$$N_{\max} = L_{2m+1} + L_{2m-1} + \cdots + L_3 + L_1 + L_{-1} + L_{-3} + \cdots + L_{-2m+1}. \tag{63}$$

For the odd indices $i = 2k-1$ we have the following property for Lucas numbers [1–3]:

$$L_{-2m+1} = -L_{2m-1}. \tag{64}$$

Taking into consideration the property (64) we can find the following value of the sum (63):

$$N_{\max} = L_{2m+1}. \tag{65}$$

Taking into consideration (62) and (65) we can formulate the following theorem.

Theorem 5. *Using* $(2m+1)$ *ternary digits in the ternary* τ*-representation* (55) *we can represent* $2L_{2m+1}+1$ *integers in the range from* $-L_{2m+1}$ *to* L_{2m+1}, *where* L_{2m+1} *is Lucas number.*

Using a result of Theorem 5 we can calculate [12] a relative code redundancy of the ternary mirror-symmetrical representation (55):

$$R = \frac{1-\log_3\tau}{\log_3\tau} = 1.283 = 128.3\%. \tag{66}$$

Thus, the relative redundancy of the ternary mirror-symmetrical number system is sufficiently large. However, this deficiency is compensated by some interesting possibilities, such as checking "golden" number representations and arithmetical operations over them.

10. The ternary mirror-symmetrical arithmetic

10.1. Mirror-symmetrical addition and subtraction

The following identities for the golden ratio powers underlie mirror-symmetrical addition:

$$2\tau^{2k} = \tau^{2(k+1)} - \tau^{2k} + \tau^{2(k-1)}, \tag{67}$$

$$3\tau^{2k} = \tau^{2(k+1)} + 0 + \tau^{2(k-1)}, \tag{68}$$

$$4\tau^{2k} = \tau^{2(k+1)} + \tau^{2k} + \tau^{2(k-1)}, \tag{69}$$

where $k = 0, \pm1, \pm2, \pm3, \ldots$

Identity (67) is a mathematical basis for the mirror-symmetrical addition of two single-digit ternary digits and gives the rule of carry realization (Table 7).

Table 7
A rule of mirror-symmetrical addition

b_k	a_k		
	$\bar{1}$	0	1
$\bar{1}$	$\bar{1}\,1\,\bar{1}$	$\bar{1}$	0
0	$\bar{1}$	0	1
1	0	1	$1\,\bar{1}\,1$

A principal peculiarity of Table 7 is the addition rule for two ternary 1's with equal signs, i.e.

$$\begin{array}{ccccccc} a_k & + & b_k & = & c_k & s_k & c_k \\ 1 & + & 1 & = & 1 & \bar{1} & 1 \\ \bar{1} & + & \bar{1} & = & \bar{1} & 1 & \bar{1} \end{array}$$

We can see that for the mirror-symmetrical addition of the ternary 1's with the same sign we have the following peculiarity. For this case it arises the intermediate sum s_k with the opposite sign and the carry c_k with the same sign. However, the carry from the kth digit spreads simultaneously to the next two digits, namely to the next left-hand, i.e $(k+1)$th digit, and to the next right-hand, i.e. $(k-1)$th digit.

Table 7 describes an operation of the simplest adder called a *single-digit half-adder*. The latter is a combinative logical circuit that has two ternary inputs a_k and b_k and two ternary outputs s_k and c_k and functions in accordance with Table 7 (Fig. 1a).

As the carrying over from the kth digit spreads to the left-hand and to the right-hand digits, this means that the full mirror-symmetrical single-digit adder has to have two inputs for the carries entering from the $(k-1)$th and $(k+1)$th digits into kth digit. Thus the full mirror-symmetrical single-digit adder is a combinative logical circuit that has 4 ternary inputs and 2 ternary outputs (Fig. 1b). Let us denote by 2Σ the mirror-symmetrical single-digit half-adder having 2 inputs and by 4Σ the mirror-symmetrical single-digit full adder having 4 inputs.

Let us describe the logical operation of the mirror-symmetrical full single-digit adder 4Σ. First of all, note that the number of all the possible 4-digit ternary input combinations of the mirror-symmetrical full adder is equal to $3^4 = 81$. The values of the output variables s_k and c_k are some discrete functions of the algebraic sum S of the input ternary variables a_k, b_k, c_{k-1}, c_{k+1}, i.e.

$$S = a_k + b_k + c_{k-1} + c_{k+1}. \tag{70}$$

Sum (70) takes the values from the set $\{-4, -3, -2, -1, 0, 1, 2, 3, 4\}$. The operation rule of the mirror-symmetrical full adder 4Σ consists in the following. The adder forms the output ternary code combination $c_k s_k$ in accordance with the value of the sum (70), i.e.

$$-4 = \bar{1}\,\bar{1}; \quad -3 = \bar{1}\,0; \quad -2 = \bar{1}\,1; \quad -1 = 0\,\bar{1}; \quad 0 = 0\,0; \quad 1 = 0\,1; \quad 2 = 1\,\bar{1}; \quad 3 = 1\,0; \quad 4 = 1\,1.$$

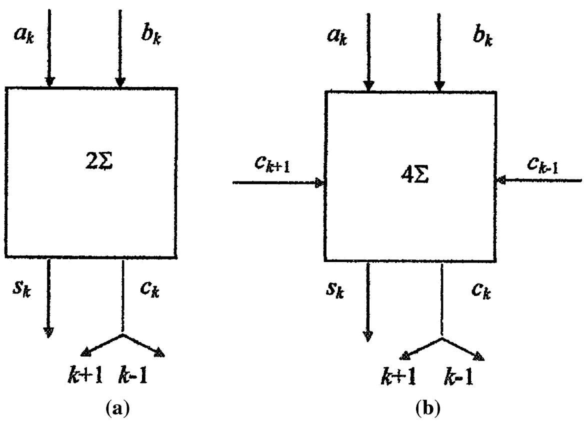

Fig. 1. Mirror-symmetrical single-digit adders: (a) a half-adder; (b) a full adder.

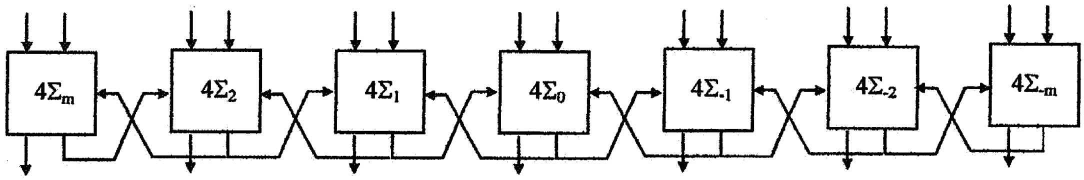

Fig. 2. Ternary mirror-symmetrical multi-digit adder.

The lower digit of such 2-digit ternary representation is the value of the intermediate sum s_k and the higher digit is the value of the carry c_k, which spreads to the next (to the left-hand and to the right-hand) digits.

The multi-digit combinative mirror symmetrical adder, which carries out the addition of two $(2m+1)$-digit mirror-symmetrical numbers, is a combinative logical circuit that consists of $2m+1$ ternary mirror-symmetrical full single-digit adders 4Σ (Fig. 2).

As an example let us consider the addition of two numbers $5+10$ in the ternary τ-system

$$\begin{array}{rcccccccc}
5 & = & 0 & 1 & \bar{1} & 1, & \bar{1} & 1 & 0 \\
10 & = & 0 & 1 & 1 & 0, & 1 & 1 & 0 \\
 & & 0 & 1 & 0 & 1, & 0 & 1 & 0 \\
 & & 1 & \leftrightarrow & 1 & & 1 & \leftrightarrow & 1 \\
\hline
5 & = & 1 & \bar{1} & 1 & 1, & 1 & \bar{1} & 1
\end{array}$$

Note the sign $\leftrightarrow$ marks the process of carry spreading.

As it is noted above, the important advantage of the ternary τ-system is a possibility to add all integers (positive and negative) in "direct" code, i.e. without the use of notions of inverse and additional codes. As an example, let us consider the addition of the negative number (-24) with the positive number 15:

$$\begin{array}{rccccccccc}
-24 & = & \bar{1} & \bar{1} & 0 & 1, & 0 & \bar{1} & \bar{1} & \\
15 & = & 1 & \bar{1} & 1 & 1, & 1 & \bar{1} & 1 & \\
 & & 0 & 1 & 1 & \bar{1}, & 1 & 1 & 0 & \\
 & & & \downarrow & 1 & \leftrightarrow & 1 & \downarrow & & \\
 & & \bar{1} & \leftrightarrow & \bar{1} & & \bar{1} & & \leftrightarrow & \bar{1} \\
-9 & = & \bar{1} & 1 & 1 & \bar{1}, & 1 & 1 & \bar{1} &
\end{array}$$

Subtraction of two mirror-symmetrical numbers $N_1 - N_2$ reduces to the addition if we represent their difference in the following form:

$$N_1 - N_2 = N_1 + (-N_2). \tag{71}$$

It follows from (71) that before the subtraction it is necessary to take the ternary inversion of the number N_2.

10.2. Mirror-symmetrical multiplication

The following trivial identity for the golden ratio powers underlies mirror-symmetrical multiplication:

$$\tau^{2n} \times \tau^{2m} = \tau^{2(n+m)}. \tag{72}$$

A rule for the mirror-symmetrical multiplication of two single digits is given in Table 8.

Multiplication is performed in "direct" code. The general algorithm of multiplication of the two multi-digit mirror-symmetrical numbers reduces to the forming of the partial products in accordance with Table 8 and their addition in accordance with the rule of the mirror-symmetrical addition. For example, let us multiply the negative number $-6 = \bar{1}\,0\,1,\,0\,\bar{1}$ by the positive number $2 = 1\,\bar{1},\,1$

$$\begin{array}{ccccccc}
 & \bar{1} & 0 & 1, & 0 & \bar{1} & \\
 & & 1 & \bar{1}, & 1 & & \\
\hline
 & & \bar{1} & 0, & 1 & 0 & \bar{1} \\
 & 1 & 0 & \bar{1}, & 0 & 1 & \\
\bar{1} & 0 & 1 & 0, & \bar{1} & & \\
\hline
\bar{1} & 1 & 0 & \bar{1}, & 0 & 1 & \bar{1}
\end{array}$$

Table 8
A rule of mirror-symmetrical multiplication

b_k	a_k		
	$\bar{1}$	0	1
$\bar{1}$	1	0	$\bar{1}$
0	0	0	0
1	$\bar{1}$	1	1

The multiplication result in the above-considered example is formed as the sum of three partial products. The first partial product $\bar{1}\,0,\ 1\,0\,\bar{1}$ is a result of the multiplication of the mirror-symmetrical number $-6 = \bar{1}\,0\,1,\ 0\,\bar{1}$ by the lowest positive unit of the mirror-symmetrical number $2 = 1\,\bar{1},\ 1$; the second partial product $1\,0\,\bar{1},\ 0\,\bar{1}$ is a result of the multiplication of the same number $-6 = \bar{1}\,0\,1,\ \ \bar{1}$ by the middle negative unit of the number $2 = 1\,\bar{1},\ 1$; and finely the third partial product $\bar{1}\,0\,1,\ 0\,\bar{1}$ is a result of the multiplication of the same number $-6 = \bar{1}\,0\,1,\ 0\,\bar{1}$ by the higher positive unit of the number $2 = 1\,\bar{1},\ 1$.

Note that the product $-12 = \bar{1}\,1\,0\,\bar{1},\ 0\,\bar{1}\,\bar{1}$ is represented in the mirror-symmetrical form. As its higher digit is a negative unit $\bar{1}$, it means that the product is a negative number.

As is shown in [12], the mirror-symmetrical division is performed in accordance with the rule of division of the classical ternary symmetrical number system. The general algorithm of the mirror-symmetrical division reduces to the sequential subtraction of a shifted divisor, which is multiplied by the next ternary numeral of the quotient.

As is shown in [12], the above ternary mirror-symmetrical arithmetic can be used as a source of new computer projects. The main property of the ternary mirror-symmetrical computer is a possibility to check all arithmetical operations.

11. Conclusion

A rather unusual approach to the fundamentals of *number theory and arithmetic* is developed in this paper. The traditional approach is that the number theory as a mathematical discipline originated in ancient Greece. However, the study of mathematics history shows that long before the Greek science a number of the outstanding discoveries, which concern number theory and arithmetic, arose in mathematics. A discovery of the positional principle of number representation is the most important achievement of the "pre-Greek" stage in mathematics history. This discovery was made by the Babylonian (present day Inale) mathematicians who used it in their sexagecimal positional number system. All the most known positional number systems, including the decimal system, which was discovered by the Hindu mathematicians presumably in the 5–8th century, and also the binary system, which underlies modern computers, are based on this principle.

It is impossible to overestimate the influence of the positional principle and the decimal system on the development of material culture. The outstanding Russian mathematician Ostrogradsky (1801–1862), who was a member of the Petersburg Academy of Sciences and many foreign academies, wrote the following:

> *It seems to us that after the invention of writing language the use by humanity of the so-called decimal notation was the greatest discovery. We want to say that the agreement, according to which we can express all useful numbers by the twelve words and their endings, is one of the most remarkable creations of human genius...*

The famous French mathematician Pierre-Simon Laplace (1749–1827), who was a member of the Parisian Academy of Sciences and honorable foreign member of the Petersburg Academy of Sciences expressed his admiration about the decimal number system in the following words:

> *The idea to represent all numbers using 9 numerals, giving to them, apart from the value by the form, another value by the position too, it seems so simple, what because of this simplicity it is difficult to understand how this idea is surprising. How was not easy to find this method, we can see on the example of the greatest geniuses of Greek science Archimedes and Apollonius, for whom this idea was remained latent.*

Why number systems, in number theory and theoretical arithmetic were not given proper attention, which they undoubtedly deserved? Possibly, the mathematical "tradition" played here a negative role. In the Greek mathematics, which reached a high standard of its development, for the first time it appeared a division of mathematics on the "higher" mathematics, to which geometry and number theory related, and the "logistic", the applied science about methods of arithmetic calculations (the "school" arithmetic), geometrical measurements and constructions. Since ancient Greeks number theory started to develop as a "pure" mathematical theory ("science for the science") that had a far reaching relation to computational practice.

Since Plato's time the "logistic" was slighted as the lowest, applied discipline, which was not included in the education of philosophers and scientists. The scornful relation to the "school" arithmetic and its problems, going back to Plato, and also an absence of any sufficient serious need for the creation of new number systems in calculation practice, which during the last centuries was entirely satisfied with the decimal system, and in the last decades with the binary system (in computer science), can explain the fact, why in number theory it was not given proper attention to the number systems and why in this part a number theory remained at the level of the Babylonian or Hindi mathematics.

The situation sharply changed in the second half of the 20th centuries after the occurrence of modern computers. In this area, there appeared again a huge interest in methods of number representation and new computer arithmetics. Except for the use of the binary number system ("Neumann's architecture"), already at the initial stage of the "computer era" the attempts to use in computers other number systems, which are distinct from the binary number system, was undertaken. During this period the number system with the "exotic" names and properties appeared: *system of residual classes, ternary symmetrical number system, number system with the complex base, nega-positional, factorial, binomial number systems*, etc. All of them had those or other advantages in comparison to the binary system and were directed to the improvement of those or other characteristics of binary computers; some of them became a basis for creation of new computer projects. In this respect, the ternary computer "Setun" designed by the Russian engineer Nikolay Brousentsov is the brightest example [12].

However, as some historical sources testify, the golden section was discovered by Babylonian mathematicians too. During many millennia two outstanding Babylonian mathematical discoveries, the positional principle of number representation and the golden section, were developing independently. Their union was realized in 1957 when the young (12-year) American mathematician George Bergman published his paper "A number system with an irrational base" [32]. In Bergman's number system [32] the golden proportion, which is an irrational number, plays the role of the beginning of all numbers because all numbers including natural, rational and irrational numbers can be represented in Bergman's number system. This result has a great methodological importance for mathematics and general science. As all numbers can be represented in Bergman's number system, that is, they reduce to the golden proportion, it means that we can formulate a new scientific doctrine "Everything is the golden proportion" instead of the Pythagorean doctrine "Everything is a number".

There is a "strange tradition" in mathematics. The history of mathematics shows that many mathematicians are not able to assess properly the outstanding mathematical discoveries of their contemporaries. As a rule, all outstanding mathematical discoveries meet incomprehension and even decisions in the moment of their occurrence and their recognition and general ecstasy start only 40–50 years after their discoveries. The 19th century became especially abundant for such blunders. Non-recognition of Lobachevsky's geometry by the Russian academic science of the 19th century, sad fate of the mathematical discoveries of the French mathematician Evariste Galois, who was killed in a duel at the age of 21 years, and the Norwegian mathematician Niels Abel, who died in poverty and obscurity at the age of 27 years, are the best examples of this "strange tradition". Unfortunately, we have to stress that the mathematical discovery of George Bergman did not have proper recognition in modem mathematics. Taking into consideration that about 50 years have passed after Bergman's discovery according to the "mathematical tradition" we have a full right to assess properly Bergman's discovery. *This discovery changes a correlation between rational and irrational numbers and puts forward the golden proportion on the first place in mathematics!* A discovery of Bergman's number system [32] and its generalizations, the Codes of the Golden Proportion [6], can be considered as a significant event in this field of knowledge. *Possibly, the number systems with irrational radices [6,32] are the most important mathematical discovery in the field of number systems after the discovery of the positional principle of number representation (Babylon, 2000 BC) and the decimal number system (India, 5–8th century)*. However, this discovery is of great importance not only for the computer practice, but also for theoretical arithmetics. As is shown in [14] and also in this paper, Bergman's number system and the codes of the golden p-proportions can become a source of new and fruitful ideas in the development of theoretical arithmetics (*Z-property of natural numbers, F- and L-codes* and so on). They allow mathematicians to give a new "constructive" definition of real number, which can become a source of new number-theoretical results. *A new theory of real numbers based on the golden p-proportions takes its origin in the Babylonian mathematics (a positional principle of number representation and the golden section), that is, a new number theory is older in its origin than the classical number theory approximately on 2000 years*. On the other hand, as is shown in [12], a new number theory can become a source of new computer projects. Thus, a new theory of real numbers, which is developed in the present paper, on the one hand, deepens theory of real numbers, and, on the other hand, returns number theory to computational practice.

As for theoretical physics and natural sciences, we can assert that the basic, fundamental concepts of the new number theory, first of all, the golden proportion and Fibonacci numbers now play an important role. This is confirmed by the following well-known scientific facts:

1. Toward the end of the 20th century, with honor due to certain physicists, the attitude of theoretical physicists to the golden proportion and Fibonacci numbers began to change sharply. The papers [34–40] demonstrate the substantial interest of modern theoretical physicists in the golden proportion as one of the fundamental mathematical constants. The works of Shechtman, Mauldin and William, El Naschie, Vladimirov and other physicists show that it is impossible to imagine a future progress in physical and cosmological research without the golden proportion. In this respect the paper [38] is a brilliant example. In this paper Prof. El Naschie shows that the majority of fundamental physical constants can be expressed via the golden proportion. The famous Russian physicist Prof. Vladimirov (Moscow University) finishes his book "Metaphysics" [40] with the following words: "*Thus, it is possible to assert that in the theory of electroweak interactions there are relations that are approximately coincident with the "Golden Section" that play an important role in the various areas of science and art*".
2. The most important modern physical discoveries such as *quasi-crystals* (Shechtman, 1982) and *fullerenes* (Nobel Price of 1996) are based on Plato's *icosahedron* (quasi-crystals) and *Archimedes truncated icosahedron* (fullerenes) that are based on the golden proportion.
3. The *hyperbolic Fibonacci and Lucas functions* [8,13,15] are one of the most important modern mathematical discoveries that can influence on the development of theoretical physics. The first important consequence of their introduction is comprehension of the fact that the classical hyperbolic functions, which were used in mathematics and theoretical physics, are not the only mathematical model of the "hyperbolic worlds". In parallel with the hyperbolic space based on the classical hyperbolic functions (Lobachevsky's hyperbolic geometry, Minkovsky's geometry, etc.), there is the "golden" hyperbolic space based on the hyperbolic Fibonacci and Lucas functions [8,13,15]. *The "golden" hyperbolic space exists objectively and independently of our consciousness. This "hyperbolic world" persistently shows itself in Living Nature. In particular, it appears in pinecones, sunflower heads, pineapples, cacti, and inflorescences of various flowers in the form of the Fibonacci and Lucas spirals on the surfaces of these biological objects (the phyllotaxis law). Note that the hyperbolic Fibonacci and Lucas functions, which underlie phyllotaxis phenomena, are not "inventions" of Fibonacci-mathematicians because they reflect objectively the major mathematical law underlying the Living Nature geometry*.
4. Except for the hyperbolic Fibonacci and Lucas functions, which can be used for simulation of the "golden" hyperbolic space, the surface of the *Golden Shofar* [16] is a new model of the "golden" hyperbolic space. The article "*Hyperbolic Universes with a Horned Topology and the CMB Anisotropy*" [41] proved to be a great surprise for the author because it showed that *the Universe geometry, is possibly, "shofarable" by its wild topological structure as indicated by El Naschie long time age* [42].
5. As to *the Fibonacci p-numbers, the golden p-proportions*, that are fundamental notions of new number theory, it is possible to assume with confidence that they can be used for simulation of the processes proceeding in Nature. In this connection, the "golden" algebraic equations [17] are of a special interest for theoretical physics because they can become mathematical models of energetic relations in molecules of different physical substances.

These facts confirm that the new number theory based on the golden p-proportions is some "natural" number theory that is of great importance for mathematics and has direct relation to theoretical physics and theoretical natural sciences. It is clear that the present paper puts forward many methodological problems that concern mathematics, theoretical physics, and general science.

References

[1] Vorobyov NN. Fibonacci numbers. Moscow: Nauka; 1978 [in Russian].
[2] Hoggat VE. Fibonacci and Lucas numbers. Palo Alto, CA: Houghton-Mifflin; 1969.
[3] Vajda S. Fibonacci & Lucas numbers, and the golden section. Theory and applications. Ellis Horwood limited; 1989.
[4] Stakhov AP. Introduction into algorithmic measurement theory. Moscow: Soviet Radio; 1977 [in Russian].
[5] Stakhov AP. Algorithmic measurement theory. Moscow: Znanie; 1979 [in Russian].
[6] Stakhov AP. Codes of the golden proportion. Moscow: Radio and Communications; 1984 [in Russian].
[7] Stakhov AP. The golden section in the measurement theory. Int J Comput Math Appl 1989;17(4–6):613–38.
[8] Stakhov AP, Tkachenko IS. Hyperbolic Fibonacci trigonometry. J Rep Ukrainian Acad Sci 1993;208(7):9–14 [in Russian].

[9] Stakhov AP. The golden section and modern harmony mathematics. Applications of Fibonacci numbers, vol. 7. Dordrecht, Boston, London: Kluwer Academic Publishers; 1998. p. 393–9.
[10] Stakhov AP. A generalization of the Fibonacci Q-matrix. J Rep Ukrainian Acad Sci 1999(9):46–9.
[11] Stakhov A, Massingue V, Sluchenkova A. Introduction into Fibonacci coding and cryptography. Kharkov: Publisher "Osnova"; 1999. ISBN 5-7768-0638-0.
[12] Stakhov AP. Brousentsov's ternary principle, Bergman's number system and ternary mirror-symmetrical arithmetic. Comput J 2002;45(2):222–36.
[13] Stakhov AP. Hyperbolic Fibonacci and Lucas Functions: a new mathematics for the living nature. Vinnitsa: IIT; 2003, ISBN 966-8432-07-X.
[14] Stakhov AP. Generalized golden sections and a new approach to geometric definition of a number. Ukrainian Math J 2004;56(8):1143–50 [in Russian].
[15] Stakhov A, Rozin B. On a new class of hyperbolic function. Chaos, Solitons & Fractals 2004;3(2):379–89.
[16] Stakhov A, Rozin B. The golden Shofar. Chaos, Solitons & Fractals 2005;26(3):677–84.
[17] Stakhov A, Rozin B. The "golden" algebraic equations. Chaos, Solitons & Fractals 2005;27(5):1415–21.
[18] Stakhov A, Rozin B. Theory of Binet formulas for Fibonacci and Lucas p-numbers. Chaos, Solitons & Fractals 2005;27(5):1162–77.
[19] Stakhov A, Rozin B. The continuous functions for the Fibonacci and Lucas p-numbers. Chaos, Solitons & Fractals 2006;28(4):1014–25.
[20] Stakhov A. The generalized principle of the golden section and its applications in mathematics, science, and engineering. Chaos, Solitons & Fractals 2005;26(2):263–89.
[21] Stakhov A. Fundamentals of a new kind of mathematics based on the golden section. Chaos, Solitons & Fractals 2005;27(5):1124–46.
[22] Spinadel VW. La familia de números metálicos en Diseño. Primer Seminario Nacional de Gráfica Digital, Sesión de Morfología y Matemática, FADU, UBA, 11–13 Junio de 1997, vol. II, ISBN 950-25-0424-9 [in Spanish].
[23] Spinadel VW. New Smarandache sequences. In: Proceedings of the first international conference on smarandache type notions in number theory, 21–24 August 1997. Lupton: American Research Press; 1997, ISBN 1-879585-58-8. p. 81–116.
[24] Spinadel VW. From the golden mean to chaos. Buenos Aires, Argentina: Editorial Nueva Librería; 1998, ISBN 950-43-9329-1 [second edition Nobuko, 2004].
[25] Spinadel VW. In: Kim Williams (editor), The metallic means and design. Nexus II: architecture and mathematics. Edizioni dell'Erba, 1998, ISBN 88-86888-13-9.
[26] Spinadel VW. A new family of irrational numbers with curious properties. Human Math Network J 1999(19):33–7. ISSN 1065-82.
[27] Spinadel VW. The metallic means family and multifractal spectra. Nonlinear Analysis 1999;36:721–45.
[28] Spinadel VW. The golden mean and its many relatives. In: First interdisciplinary conference of the international society of the arts, mathematics and architecture ISAMA 99, San Sebastián, Spain, 7–11 June 1999, p. 453–60, ISBN 84-930669-0-7.
[29] Spinadel VW. The family of metallic means. Vis Math 1999;1(3). Available from: http://members.tripod.com/vismath/.
[30] Spinadel VW. The family of metallic means. Symmetry: culture and science. Quart Int Soc Interdisciplinary Study Symm (ISIS-Symmetry) 1999;10(3–4):317–38.
[31] Spinadel VW. The metallic means family and renormalization group techniques. Proc Steklov Inst Math 2000(1):194–209.
[32] Spinadel VW. The metallic means family and forbidden symmetries. Int Math J 2002;2(3):279–88.
[33] Bergman G. A number system with an irrational base. Math Mag 1957(31):98–119.
[34] Mauldin RD, Willams SC. Random recursive construction. Trans Am Math Soc 1986;295:325–46.
[35] El Naschie MS. On dimensions of Cantor set related systems. Chaos, Solitons & Fractals 1993;3:675–85.
[36] El Nashie MS. Quantum mechanics and the possibility of a Cantorian space-time. Chaos, Solitons & Fractals 1992;1:485–7.
[37] El Nashie MS. Is quantum space a random cantor set with a golden mean dimension at the core? Chaos, Solitons & Fractals 1994;4(2):177–9.
[38] El Naschie MS. Elementary number theory in superstrings, loop quantum mechanics, twisters and E-infinity high energy physics. Chaos, Solitons & Fractals 2006;27(2):297–330.
[39] Vladimirov YS. Quark icosahedron, charges and Vainberg's angle. In: Proceedings of the international conference "problems of harmony, symmetry and the golden section in nature, science, and art. Vinnitsa: Vinnitsa State Agricultural University; 2003. p. 69–79.
[40] Vladimirov YS. Metaphysics. Moscow: Binom; 2002.
[41] Aurich R, Lusting S, Steiner F, Then H. Hyperbolic universes with a horned topology and the CMB anisotropy. Classical Quantum Gravity 2004(21):4901–26.
[42] El Naschie MS. Wild topology, hyperbolic geometry and fusion algebra of high energy particles physics. Chaos, Solitons & Fractals 2002;13(9):1932–46.