THE THIRD AND FOURTH CONSTANTS OF SMARANDACHE

by

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In the present note we prove the divergence of some series involving the Smarandache function, using an unitary method, and then we prove that the series

$$\sum_{n=2}^{\infty} \frac{1}{S(2)S(3)\ldots S(n)}$$

is convergent to a number $s \in (71/100,\ 101/100)$ and we study some applications of this series in the Number Theory (third constant of Smarandache).

The Smarandache Function $S : N^* \to N$ is defined [1] such that $S(n)$ is the smallest integer k with the property that k! is divisible by n.

Proposition 1. If $(x_n)_{n \geq 1}$ is a strict increasing sequence of natural numbers, then the series :

$$\sum_{n=1}^{\infty} \frac{x_{n+1} - x_n}{S(x_n)}, \tag{1}$$

where S is the Smarandache function, is divergent.

Proof. We consider the function $f : [x_n, x_{n+1}] \to \mathbf{R}$, defined by $f(x) = \ln \ln x$. It fulfils the conditions of the Lagrange's theorem of finite increases. Therefore there is $c_n \in (x_n, x_{n+1})$ such that :

$$\ln \ln x_{n+1} - \ln \ln x_n = \frac{1}{c_n \ln c_n}(x_{n+1} - x_n). \tag{2}$$

Because $x_n < c_n < x_{n+1}$, we have :

$$\frac{x_{n+1} - x_n}{x_{n+1} \ln x_{n+1}} < \ln \ln x_{n+1} - \ln \ln x_n < \frac{x_{n+1} - x_n}{x_n \ln x_n}, \quad (\forall) n \in N, \tag{3}$$

if $x_n \neq 1$.

We know that for each $n \in N^* \setminus \{1\}$, $\frac{S(n)}{n} \le 1$, i.e.

$$0 < \frac{S(n)}{n \ln n} \le \frac{1}{\ln n}, \tag{4}$$

from where it results that $\lim_{n \to \infty} \frac{S(n)}{n \ln n} = 0$. Hence there exists $k > 0$ such that $\frac{S(n)}{n \ln n} < k$, i.e., $n \ln n > \frac{S(n)}{k}$ for any $n \in N^*$, so

$$\frac{1}{x_n \ln x_n} < \frac{k}{S(x_n)}. \tag{5}$$

Introducing (5) in (3) we obtain :

$$\ln \ln x_{n+1} - \ln \ln x_n < k \frac{x_{n+1} - x_n}{S(x_n)}, \ (\forall) n \in N^* \setminus \{1\}. \tag{6}$$

Summing up after n it results :

$$\sum_{n=1}^{m} \frac{x_{n+1} - x_n}{S(x_n)} > \frac{1}{k}(\ln \ln x_{m+1} - \ln \ln x_1).$$

Because $\lim_{m \to \infty} x_m = \infty$ we have $\lim_{m \to \infty} \ln \ln x_m = \infty$, i.e., the series :

$$\sum_{n=1}^{\infty} \frac{x_{n+1} - x_n}{S(x_n)}$$

is divergent. The Proposition 1 is proved.

Proposition 2. Series $\sum_{n=2}^{\infty} \frac{1}{S(n)}$, where S is the Smarandache function, is divergent.

Proof. We use Proposition 1 for $x_n = n$.

Remarks. 1) If x_n is the n - th prime number, then the series $\sum_{n=1}^{\infty} \frac{x_{n+1} - x_n}{S(x_n)}$ is divergent.

2) If the sequence $(x_n)_{n \ge 1}$ forms an arithmetical progression of natural numbers, then the series $\sum_{n=1}^{\infty} \frac{1}{S(x_n)}$ is divergent.

3) The series $\sum_{n=1}^{\infty} \frac{1}{S(2n+1)}$, $\sum_{n=1}^{\infty} \frac{1}{S(4n+1)}$ etc., are all divergent.

In conclusion, Proposition 1 offers us an unitary method to prove that the series having one of the precedent aspects are divergent.

Proposition 3. The series :

$$\sum_{n=2}^{\infty} \frac{1}{S(2)\cdot S(3)\cdots S(n)},$$

where S is the Smarandache function, is convergent to a number s ∈ (71/100, 101/100).

Proof. From the definition of the Smarandache function it results $S(n) \leq n!$, $(\forall)n \in N^*\setminus\{1\}$, so $\frac{1}{S(n)} \geq \frac{1}{n!}$.

Summing up, begining with n = 2 we obtain :

$$\sum_{n=2}^{\infty} \frac{1}{S(2)\cdot S(3)\cdots S(n)} \geq \sum_{n=2}^{\infty} \frac{1}{n!} = e-2.$$

The product S(2) · S(3) ... S(n) is greater than the product of prime numbers from the set {1, 2, ..., n}, because S(p) = p, for p = prime number. Therefore :

$$\frac{1}{\prod_{i=2}^{n} S(i)} < \frac{1}{\prod_{i=1}^{k} p_i}, \tag{7}$$

where p_k is the biggest prime number smaller or equal to n.

There are the inequalities :

$$S = \sum_{n=2}^{\infty} \frac{1}{S(2)S(3)\cdots S(n)} = \frac{1}{S(2)} + \frac{1}{S(2)S(3)} + \frac{1}{S(2)S(3)S(4)} + \cdots +$$

$$+\frac{1}{S(2)S(3)\cdots S(k)} + \cdots < \frac{1}{2} + \frac{2}{2\cdot 3} + \frac{2}{2\cdot 3\cdot 5} + \frac{4}{2\cdot 3\cdot 5\cdot 7} +$$

$$+\frac{2}{2\cdot 3\cdot 5\cdot 7\cdot 11} + \cdots + \frac{p_{k+1} - p_k}{p_1 p_2 \cdots p_k} + \cdots. \tag{8}$$

Using the inequality $p_1 p_2 \cdots p_k > p_{k+1}^3$, $(\forall)k \geq 5$ [2], we obtain :

$$S < \frac{1}{2} + \frac{1}{3} + \frac{1}{15} + \frac{2}{105} + \frac{1}{p_6^2} + \frac{1}{p_7^2} + \cdots + \frac{1}{p_{k+1}^2} + \cdots. \qquad (9)$$

We note $P = \frac{1}{p_6^2} + \frac{1}{p_7^2} + \cdots$ and observe that $P < \frac{1}{13^2} + \frac{1}{14^2} + \frac{1}{15^2} + \cdots$.

It results :

$$P < \frac{\pi^2}{6} - \left(1 + \frac{1}{2^2} + \frac{1}{3^2} + \cdots + \frac{1}{12^2}\right),$$

where

$$\frac{\pi^2}{6} = 1 + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \cdots \quad \text{(EULER)}.$$

Introducing in (9) we obtain :

$$S < \frac{1}{2} + \frac{1}{3} + \frac{1}{15} + \frac{2}{105} + \frac{\pi^2}{6} - 1 - \frac{1}{2^2} - \frac{1}{3^2} - \cdots - \frac{1}{12^2}.$$

Estimating with an approximation of an order not more than $\frac{1}{10^2}$, we find :

$$0,71 < \sum_{n=2}^{\infty} \frac{1}{S(2)S(3)\cdots S(n)} < 1,01. \qquad (10)$$

The Proposition 3 is proved.

Remark. Giving up at the right increase from the first terms in the inequality (8) we can obtain a better right framing :

$$\sum_{n=2}^{\infty} \frac{1}{S(2)S(3)\cdots S(n)} < 0,97. \qquad (11)$$

Proposition 4. Let α be a fixed real number, $\alpha \geq 1$. Then the series $\sum_{n=2}^{\infty} \frac{n^{\alpha}}{S(2)S(3)\cdots S(n)}$ is convergent (fourth constant of Smarandache).

Proof. Be $(p_k)_{k \geq 1}$ the sequence of prime numbers. We can write :

$$\frac{2^\alpha}{S(2)} = \frac{2^\alpha}{2} = 2^{\alpha-1}$$

$$\frac{3^\alpha}{S(2)S(3)} = \frac{3^\alpha}{p_1 p_2}$$

$$\frac{4^\alpha}{S(2)S(3)S(4)} < \frac{4^\alpha}{p_1 p_2} < \frac{p_3^\alpha}{p_1 p_2}$$

$$\frac{5^\alpha}{S(2)S(3)S(4)S(5)} < \frac{5^\alpha}{p_1 p_2 p_3} < \frac{p_4^\alpha}{p_1 p_2 p_3}$$

$$\frac{6^\alpha}{S(2)S(3)S(4)S(5)S(6)} < \frac{6^\alpha}{p_1 p_2 p_3} < \frac{p_4^\alpha}{p_1 p_2 p_3}$$

....

$$\frac{n^\alpha}{S(2)S(3)\cdots S(n)} < \frac{n^\alpha}{p_1 p_2 \cdots p_k} < \frac{p_{k+1}^\alpha}{p_1 p_2 \cdots p_k},$$

where $p_i \le n$, $i \in \{1, ..., k\}$, $p_{k+1} > n$.

Therefore

$$\sum_{n=2}^{\infty} \frac{n^\alpha}{S(2)S(3)\cdots S(n)} < 2^{\alpha-1} + \sum_{k=1}^{\infty} \frac{(p_{k+1} - p_k) \cdot p_{k+1}^\alpha}{p_1 p_2 \cdots p_k} <$$

$$< 2^{\alpha-1} + \sum_{k=1}^{\infty} \frac{p_{k+1}^{\alpha+1}}{p_1 p_2 \cdots p_k}.$$

Then it exists $k_0 \in \mathbb{N}$ such that for any $k \ge k_0$ we have :

$$p_1 p_2 \cdots p_k > p_{k+1}^{\alpha+3}.$$

Therefore

$$\sum_{n=2}^{\infty} \frac{n^\alpha}{S(2)S(3)\cdots S(n)} < 2^{\alpha-1} + \sum_{k=1}^{k_0-1} \frac{p_{k+1}^{\alpha+1}}{p_1 p_2 \cdots p_k} + \sum_{k \ge k_0} \frac{1}{p_{k+1}^2}$$

Because the series $\sum_{k \ge k_0} \frac{1}{p_{k+1}^2}$ is convergent it results that the given series is convergent too.

Consequence 1. It exists $n_0 \in N$ so that for each $n \ge n_0$ we have $S(2)S(3) \ldots S(n) > n^\alpha$.

Proof. Because $\lim_{n \to \infty} \frac{n^\alpha}{S(2)S(3)\cdots S(n)} = 0$, there is $n_0 \in N$ so that

$$\frac{n^\alpha}{S(2)S(3)\cdots S(n)} < 1 \text{ for each } n \ge n_0.$$

Consequence 2. It exists $n_0 \in N$ so that :

$$S(2) + S(3) + \cdots + S(n) > (n-1) \cdot n^{\frac{\alpha}{n-1}} \text{ for each } n \ge n_0.$$

Proof. We apply the inequality of averages to the numbers S(2), S(3), ..., S(n) :

$$S(2) + S(3) + \cdots + S(n) > (n-1) \sqrt[n-1]{S(2)S(3)\cdots S(n)} > (n-1) n^{\frac{\alpha}{n-1}}, \ \forall n \ge n_0.$$

REFERENCES

[1] **E. Burton** : *On some series involving the Smarandache Function*, Smarandache Function Journal, vol. 6, N° 1 (1995), 13-15.

[2] **L. Panaitopol** : *Asupra unor inegalitati ale lui Bonse*, Gazeta Matematica, seria A, vol. LXXVI, nr. 3, 1971, 100 - 102.

[3] **F. Smarandache** : *A Function in the Number Theory* (An. Univ. Timisoara, Ser. St. Mat., vol. XVIII, fasc. 1 (1980), 79-88).

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