A BASIC CHARACTERISTIC OF TWIN PRIMES AND ITS GENERALIZATION

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ABSTRACT. *The sum of powers of positive divisors of an integer, expressed in terms of the floor function, provides the basis for another characterization of twin primes in particular, and of prime k-tuples generally. This elementary characterization is deployed in a software test for prime k-tuples using Mathematica®.*

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Introduction

Prime numbers [6] are integers > 1 divisible only by unity and itself. Thus, *2, 3, 5, 7, 11, 13, 17, 19, 23, 29, 31, 37, 41, ...* are the first few primes. And, twin primes [12] are those pairs of primes, like *(5, 7)* or *(13001, 13003)*, differing by *2*.

There are exactly *27 412 679* such twins up to ten billion compared to *455 052 511* individual primes below the same limit. The largest known twin primes are $665\,551\,035 \cdot 2^{80025} \pm 1$, each of *24 099* digits, discovered by *David Underbakke* and *Yves Gallot* [3] on November 28, 2000.

What condition is necessary for a number pair to be twin primes ? In 1949, *P A Clement* [4] characterized twin primes by proving that for $n \geq 2$, the pair $(n, n+2)$ of integers are twin primes if and only if $4[(n-1)!+1]+n \equiv 0 \mod n(n+2)$. Unfortunately, this test has no practical application due to the high cost of computing the factorial function.

By comparison, the following alternative characterization, found by *Ruiz* in 2000 and reported by *Eric W Weisstein* [17] on the Internet, is computationally friendlier.

Theorem 1 For $a \geq 0$, the pair $(n, n+2)$ of integers are twin primes if and only if

$$\sum_{i=1}^{n} i^a\left(\left\lfloor \frac{n+2}{i} \right\rfloor + \left\lfloor \frac{n}{i} \right\rfloor\right) = 2 + n^a + \sum_{i=1}^{n} i^a\left(\left\lfloor \frac{n+1}{i} \right\rfloor + \left\lfloor \frac{n-1}{i} \right\rfloor\right)$$

where $\lfloor x \rfloor$ is the floor function [8] [9] denoting the greatest integer not exceeding x.

This article provides a proof of the above result, its generalization to other prime k-tuples, and the Mathematica® [16] [18] code for implementing the k-tuple primality test.

Preliminaries

This article is dependent on the following simple fact published in the following article of The Smarandache Notion Journal: [14] and seldom explicitly mentioned in standard texts on number theory. Known exceptions are those by *Trygve Nagell* [11] and *David M Burton* [2].

Lemma 1 For $n > 0$, $\left\lfloor \frac{n}{i} \right\rfloor - \left\lfloor \frac{n-1}{i} \right\rfloor = 1$ or 0 according as i divides n or not.

Proof We recall the division algorithm [15] which states that for any integers n and i, with i positive, there are unique integers q *(quotient)* and r *(remainder)* such that $n = qi+r$, where $i > r \geq 0$.

By the division algorithm, if $i \mid n$ then $r = 0$ giving $\lfloor \frac{n}{i} \rfloor = q$, $\lfloor \frac{n-1}{i} \rfloor = q-1$. Otherwise, $i > r > 0$ giving $\lfloor \frac{n}{i} \rfloor = \lfloor \frac{n-1}{i} \rfloor = q$. ■

We now consider the two arithmetic functions [10] $\tau(n)$ and $\sigma_a(n)$ which are intimately related to the above property. The divisor function $\tau(n)$, the number of positive divisors of n, is expressed as $\sum_{d|n} 1$, while the sum $\sigma_a(n)$ of the a^{th} powers of the positive divisors of n can be written as $\sum_{d|n} d^a$. Thus, $\tau(n) = \sigma_0(n)$ and Lemma 1 implies the relationships:

$$\tau(n) = \sum_{i=1}^{n} \left(\left\lfloor \frac{n}{i} \right\rfloor - \left\lfloor \frac{n-1}{i} \right\rfloor \right) \tag{1}$$

$$\sigma_a(n) = \sum_{i=1}^{n} i^a \left(\left\lfloor \frac{n}{i} \right\rfloor - \left\lfloor \frac{n-1}{i} \right\rfloor \right) \tag{2}$$

In what follows, the arithmetic functions $\tau(n)$ and $\sigma_a(n)$ shall be defined only for positive values of their arguments. And, 1 is neither prime nor composite.

Defining ***proper*** divisors of n as those excluding 1 and n, we derive a more efficient version of relation (1) with minimal change.

Lemma 2 For n > 1, $\tau(n) = 2 + \sum_{i=2}^{j} \left(\left\lfloor \frac{n}{i} \right\rfloor - \left\lfloor \frac{n-1}{i} \right\rfloor \right)$ where j is the highest proper divisor of n, the summation being 0 if j is nonexistent.

Proof Clearly, none of $j+1, j+2, \ldots, n-1$ are divisors of n and the constant 2 accounts for the cases $i = 1$ and $i = n$ which are not proper divisors of n. ■

In general, it is sufficient to assume that j is $\lfloor \frac{n}{2} \rfloor$ or $\lfloor \frac{n}{3} \rfloor$ according as n is even or odd. In particular, it may be possible to choose the parity of i for specific cases of n. Applying such resources on Theorem 1, we readily obtain the example:

Corollary 1 For odd $n > 7$, the pair $(n, n+2)$ of integers are twin primes if and only if

$$\sum_{i \text{ odd}}^{j} \left(\left\lfloor \frac{n+2}{i} \right\rfloor - \left\lfloor \frac{n+1}{i} \right\rfloor + \left\lfloor \frac{n}{i} \right\rfloor - \left\lfloor \frac{n-1}{i} \right\rfloor \right) = 2$$

where the summation is over odd values of i through $j = \lfloor \frac{n}{3} \rfloor$.

We next establish two lemmas, including an extended expression for $\sigma_a(n+e)$, which will become useful in proving Theorem 1 and its generalization.

Lemma 3 If $a \geq 0$ and $(n+2) > e > 0$, then

$$\sigma_a(n+e) = \sum_{i=1}^{n} i^a \left(\left\lfloor \frac{n+e}{i} \right\rfloor - \left\lfloor \frac{n+e-1}{i} \right\rfloor \right) + (n+e)^a$$

Proof Clearly $(n+2) > e > 0 \rightarrow 2 > \frac{n+e}{n+1} \geq 1 \rightarrow \left\lfloor \frac{n+e}{n+j} \right\rfloor = 1$ for $1 \leq j \leq e$

so that $\sum_{i=n+1}^{n+e} i^a \left(\left\lfloor \frac{n+e}{i} \right\rfloor - \left\lfloor \frac{n+e-1}{i} \right\rfloor \right) = \sum_{i=n+1}^{n+e} i^a - \sum_{i=n+1}^{n+e-1} i^a = (n+e)^a$.

Thus $\sigma_a(n+e) = \sum_{i=1}^{n+e} i^a \left(\left\lfloor \frac{n+e}{i} \right\rfloor - \left\lfloor \frac{n+e-1}{i} \right\rfloor \right)$ by (2)

$$= \sum_{i=1}^{n} i^a \left(\left\lfloor \frac{n+e}{i} \right\rfloor - \left\lfloor \frac{n+e-1}{i} \right\rfloor \right) + (n+e)^a \quad \blacksquare$$

Lemma 4 For a set $\{1, m_1, m_2, \dots, m_k\}$ of positive integers,

$$\sum_{i=1}^{k} \sigma_a(m_i) = k + \sum_{i=1}^{k} m_i^a$$

if and only if $m_1, m_2, \dots, m_k$ are all primes.

Proof The condition in the lemma is evidently sufficient. To prove equivalence, we note that

$$\sigma_a(m_i) \geq 1 + m_i^a$$

by counting only the non-proper divisors of m_i and therefore

$$\sum \sigma_a(m_i) \geq \sum 1 + \sum m_i^a \tag{3}$$

over equal summation limits.

Without loss of generality, suppose now that $\sigma_a(m_1) > 1 + m_1^a$,

that is $$k + \sum_{i=1}^{k} m_i^a - \sum_{i=2}^{k} \sigma_a(m_i) > 1 + m_1^a$$

or $$\sum_{i=2}^{k} \sigma_a(m_i) < (k-1) + \sum_{i=2}^{k} m_i^a$$

which contradicts (3), and therefore $\sigma_a(m_1) = 1 + m_1^a$.

Hence m_1 is prime. Similarly, the hypothesis $\sigma_a(m_i) > 1 + m_i^a$ yields a contradiction for each other i and the result follows. $\blacksquare$

3 Proof of Theorem 1

If the pair $(n, n+2)$ of integers are twin primes, then by definition,

$$\sigma_a(n) + \sigma_a(n+2) = 2 + n^a + (n+2)^a \tag{4}$$

From (2) and Lemma 3, we also have

$$\sigma_a(n) + \sigma_a(n+2) = \sum_{i=1}^{n} i^a \left(\left\lfloor \frac{n+2}{i} \right\rfloor - \left\lfloor \frac{n+1}{i} \right\rfloor + \left\lfloor \frac{n}{i} \right\rfloor - \left\lfloor \frac{n-1}{i} \right\rfloor \right) + (n+2)^a$$

Hence $$\sum_{i=1}^{n} i^a \left(\left\lfloor \frac{n+2}{i} \right\rfloor + \left\lfloor \frac{n}{i} \right\rfloor \right) = 2 + n^a + \sum_{i=1}^{n} i^a \left(\left\lfloor \frac{n+1}{i} \right\rfloor + \left\lfloor \frac{n-1}{i} \right\rfloor \right) \tag{5}$$

Conversely, if (5) holds, then (4) is implied and Lemma 4 completes the proof. $\blacksquare$

A generalization

A set $\{e_1, e_2, \ldots, e_k\}$ of positive integers is said to be ***admissible*** if $n, n+e_1, n+e_2, \ldots, n+e_k$ is not excluded by divisibility considerations as a possible sequence of primes. Thus, $\{2, 6\}$ and $\{4, 6\}$ are admissible sets. But $\{2, 4\}$ is not, as $(n, n+2, n+4)$ is never a prime triplet when $n > 3$. *Hans Riesel* [**13**] discusses a method of determining admissible sets.

<u>**Theorem 2**</u> If $a \geq 0$, $e_0 = 0$ and $\{e_1, e_2, \ldots, e_k\}$ is an admissible set of positive integers in the open interval $(0, n-2)$, then $(n, n+e_1, n+e_2, \ldots, n+e_k)$ is a sequence of primes if and only if

$$\sum_{i=1}^{n} i^a \left(\sum_{j=0}^{k} \left\lfloor \frac{n+e_j}{i} \right\rfloor \right) = 1 + k + n^a + \sum_{i=1}^{n} i^a \left(\sum_{j=0}^{k} \left\lfloor \frac{n+e_j-1}{i} \right\rfloor \right)$$

<u>Proof</u> If $(n+e_0, n+e_1, n+e_2, \ldots, n+e_k)$ is a prime $(k+1)$-tuple, then by definition

$$\sum_{j=0}^{k} \sigma_a(n+e_j) = 1 + k + \sum_{j=0}^{k} (n+e_j)^a \tag{6}$$

From (2) and Lemma 3, we also have

$$\sum_{j=0}^{k} \sigma_a(n+e_j) = \sum_{i=1}^{n} i^a \left(\sum_{j=0}^{k} \left(\left\lfloor \frac{n+e_j}{i} \right\rfloor - \left\lfloor \frac{n+e_j-1}{i} \right\rfloor \right) \right) + \sum_{j=1}^{k} (n+e_j)^a \tag{7}$$

Equating (6) and (7) and simplifying, we obtain

$$\sum_{i=1}^{n} i^a \left(\sum_{j=0}^{k} \left(\left\lfloor \frac{n+e_j}{i} \right\rfloor - \left\lfloor \frac{n+e_j-1}{i} \right\rfloor \right) \right) = 1 + k + n^a \tag{8}$$

Conversely, if (8) holds, then (6) is implied and Lemma 4 completes the proof. ■

A variation

Theorem 2, as it stands, requires $n > \max\{e_1, e_2, \ldots, e_k\} - 2$ through its dependence on the open interval $(0, n+2)$. However, that restriction may be removed by avoiding Lemma 3 in the proof of the theorem.

<u>Proof</u> By Lemma 1, if $(n+e_j)$ is prime, then

$$\sum_{j=0}^{k} \left(\left\lfloor \frac{n+e_j}{i} \right\rfloor - \left\lfloor \frac{n+e_j-1}{i} \right\rfloor \right) = \begin{cases} 1+k & \text{if } i = 1 \\ 1 & \text{if } i = n \\ 0 & \text{otherwise} \end{cases}$$

Therefore, if $(n+e_0, n+e_1, n+e_2, \ldots, n+e_k)$ are all primes, then

$$\sum_{i=1}^{n} i^a \sum_{j=0}^{k} \left(\left\lfloor \frac{n+e_j}{i} \right\rfloor - \left\lfloor \frac{n+e_j-1}{i} \right\rfloor \right) = 1 + k + n^a \tag{9}$$

as all other terms, involving non-divisors, vanish.

However, if any one of $(n+e_0, n+e_1, n+e_2, \ldots, n+e_k)$ is composite, then by Lemma 1, (9) becomes

$$\sum_{i=1}^{n} i^a \sum_{j=0}^{k} \left(\left\lfloor \frac{n+e_j}{i} \right\rfloor - \left\lfloor \frac{n+e_j-1}{i} \right\rfloor \right) > 1 + k + n^a$$

due to a proper divisor of the composite element. Thus, equality is only possible for prime $(k+1)$-tuples. ■

Software codes

The more common methods of preparing a list of twin primes do not rely upon any test for such pairs. Instead, some sieve [5] [7] [13] method is employed to sift out all primes below a required limit and a simple search then extracts the twins.

On the other hand, given a pair $(n, n+2)$ of integers, Corollary 1 represents a possible test to simultaneously determine if they are twin primes without using a list of primes. It may not be the fastest available twin-primality test but its implementation is fairly straightforward as shown by the interactive Mathematica® dialogue:

```
In[1]:=    n = 2000081; If[Sum[Floor[(n+2)/i] - Floor[(n+1)/i]
             + Floor[n/i] - Floor[(n-1)/i], {i,1,Floor[n/3],2}]
             == 2, "True", "False"]

Out[1]=    True
```

Note that the *Floor*[x/y] function may be replaced by its equivalent *Quotient*[x, y] which is somewhat faster [1].

The following example is a non-optimum implementation of Theorem 2 with $a = 3$ to search for prime quadruplets $(n, n+2, n+6, n+8)$ below 10000.

```
In[2]:=    a = 3; n = 10000; e = {0, 2, 6, 8};
           Do[If[Sum[i^a Floor[(j + e[[k]])/i], {k, Length[e]},{i, j}]
             == Length[e] + j^a + Sum[i^a Floor[(j + e[[k]] - 1)/i],
             {k, Length[e]},{i, j}], Print[Table[j + e[[k]],
             {k, Length[e]}]]],{j, n}]

           {5, 7, 11, 13}
           {11, 13, 17, 19}
           {101, 103, 107, 109}
           {191, 193, 197, 199}
           {821, 823, 827, 829}
           {1481, 1483, 1487, 1489}
           {1871, 1873, 1877, 1879}
           {2081, 2083, 2087, 2089}
           {3251, 3253, 3257, 3259}
           {3461, 3463, 3467, 3469}
           {5651, 5653, 5657, 5659}
           {9431, 9433, 9437, 9439}
```

References

1 **David Bressoud & Stan Wagon.** *A Course in Computational Number Theory.* Key College Publishing, Emeryville, *2000.*

2 **David M Burton.** *Elementary Number Theory, Fourth Edition.* McGraw-Hill, New York, *1998.*

3 **Chris K Caldwell.** *The Top Twenty: Twin Primes.* http://www.utm.edu/research/primes/lists/top20/twin.html

4 **P A Clement.** Congruences for sets of primes. *American Mathematical Monthly,* 56:23-25, *1949.*

5 **Peter Giblin.** *Primes and Programming: An Introduction to Number Theory with Computing.* Cambridge University Press, Cambridge, *1993.*

6 **Godfrey H Hardy & Edward M Wright.** *An Introduction to the Theory of Numbers, Fifth Edition.* Clarendon Press, Oxford, *1979.*

7 **Mark Herkommer.** *Number Theory: A Programmer's Guide.* McGraw-Hill, New York, *1999.*

8 **Kenneth E Iverson.** *A Programming Language.* John Wiley, New York, *1962.*

9 **Donald E Knuth.** *The Art of Computer Programming II: Seminumerical Algorithms, Third Edition.* Addison Wesley, Reading, Massachusetts, *1998.*

10 **Calvin T Long.** *Elementary Introduction to Number Theory, Third Edition.* Waveland Press, Prospect Heights, Illinois, *1995.*

11 **Trygve Nagell.** *Introduction to Number Theory, Second Edition.* Chelsea Publishing, New York, *1981.*

12 **Paulo Ribenboim.** *The New Book of Prime Number Records, Third Edition.* Springer-Verlag, New York, *1995.*

13 **Hans Riesel.** *Prime Numbers and Computer Methods for Factorization, Second Edition.* Birkhäuser, Boston, *1994.*

14 **Sebastián M Ruiz.** A functional recurrence to obtain the prime numbers using the Smarandache prime function. *Smarandache Notions Journal,* **11**:56-58, *2000.*

15 **James J Tattersall.** *Elementary Number Theory in Nine Chapters.* Cambridge University Press, Cambridge, *1999.*

16 **Stan Wagon.** *Mathematica® in Action, Second Edition.* Springer/TELOS, New York, *2000.*

17 **Eric W Weisstein.** *Twin Primes.* http://mathworld.wolfram.com/TwinPrimes.html

18 **Stephen Wolfram.** *The Mathematica® Book: A System for Doing Mathematics by Computer, Fourth Edition.* Cambridge University Press, Cambridge, *1999.*