A Classification of s-Lines in a Closed s-Manifold

Howard Iseri
Dept. of Mathematics and Computer Information Science
Mansfield University
Mansfield, PA 16933
hiseri@mnsfld.edu

Abstract: In *Smarandache Manifolds* [1], it is shown that the s-sphere has both closed and open s-lines. It is shown here that this is true for any closed s-manifold. This would make each closed s-manifold a Smarandache geometry relative to the axiom requiring each line to be extendable to infinity, since each closed s-line would have finite length. Furthermore, it is shown that whether a particular s-line is closed or not is determined locally, and it is determined precisely which s-lines are closed and which are open.

Introduction

Recall that an s-manifold is the union of equilateral triangular disks that are identified edge to edge. Furthermore, each vertex is shared by exactly five, six, or seven distinct triangles, and each edge is shared by exactly two distinct triangles. The s-lines are defined to be curves that are as straight as possible. In particular, they are straight in a very natural sense within each triangle and across the edges. Across vertices, s-lines make two equal angles (see [1]). In general, a manifold is closed if it is compact and has no boundary, like the surface of a sphere or torus. Here, the term *closed* is used in the same way that it is used in *simple closed curve*. Since each edge is identified in an s-manifold, there is no boundary. Therefore, an s-manifold being closed is equivalent to its consisting of a finite number of triangles.

In [1], the concept of a locally linear projection was used to investigate the behavior of s-lines in the s-sphere. We will expand on that investigation here.

Locally Linear Projections

The plane can be tiled by equilateral triangles. The tiling we will use is the one that has the segment from (0,0) to (1,0) as one of the edges, and we will focus initially on the triangle that lies above this segment. A locally linear projection of an s-line l from an s-manifold is constructed as follows. Choose a segment from l that spans one of the triangles. We identify this triangle with the one that lies above the segment from (0,0) to (1,0) so that exactly one point of l lies on this segment. This can be done in several ways. We then extend this segment in one direction, exactly as it extends in the s-manifold. At a vertex, we will maintain the angle that lies to the right of the projection.

In this tiling, all points in the plane can be expressed as a linear combination of the vectors $[1,0]$ and $[1/2,\sqrt{3}/2]$. The vertices of the triangles correspond to those linear combinations with integer coefficients. The linear combination $a[1,0] + b[1/2, \sqrt{3}/2]$ has rectangular coordinates $(a+b/2, \sqrt{3}b/2)$. In rectangular coordinates, the point (x,y) corresponds to $(x-y/\sqrt{3})[1,0] + (y/\sqrt{3})[1/2, \sqrt{3}/2]$. It follows that a line from the origin to any vertex will have slope $m = \sqrt{3}b/(2a+b)$ with a and b integers. If a line from the origin has slope $m = \sqrt{3}y/x$ with x and y integers (i.e., m is a rational multiple of $\sqrt{3}$), then it will pass through the vertex $(x-y)[1,0] + y[1/2, \sqrt{3}/2]$. In other words, a line will pass through the origin and another vertex, if, and only if, it is a rational multiple of $\sqrt{3}$. Clearly, this can be extended to the following.

Lemma 1. *A line passing through a vertex will pass through another vertex, if, and only if, its slope is a rational multiple of* $\sqrt{3}$.

The locally linear projection of an s-line will change directions only at certain vertices. It is reasonable to talk about the slope m of the projection and the angle θ it makes with the positive x-axis, even though it may change from segment to segment. The relation between these is $m = \tan\theta$. When an s-line passes through an elliptic vertex (one with five triangles around it), the slope of its projection is reduced by $30°$. When it passes through a hyperbolic vertex (one with seven triangles around it), the slope of its projection is increased by $30°$. Since $\tan(\theta+30°) = \sqrt{3}(\tan\theta/\sqrt{3} + 1/3)/(1 - \tan\theta/\sqrt{3})$ and $\tan(\theta-30°) = \sqrt{3}(\tan\theta/\sqrt{3} - 1/3)/(1 + \tan\theta/\sqrt{3})$, it is clear that $\tan(\theta+30°)$ and $\tan(\theta-30°)$ will be rational multiples of $\sqrt{3}$ whenever $\tan\theta$ is. This gives us the following.

Lemma 2. *The angle of a locally linear projection of an s-line is constant modulo* $30°$, *and its slope will always be a rational multiple of* $\sqrt{3}$, *or it will always be an irrational multiple of* $\sqrt{3}$.

Classification of Closed and Open s-Lines

Given some closed s-manifold, it would seem that whether a particular s-line is closed or not would depend on the global structure of the s-manifold. We will show, however, that we can determine this by looking at a segment of the s-line in any of the triangles of the s-manifold.

Let l be an s-line. We look at a segment of it that spans some triangle, and consider a locally linear projection λ based on this segment. The slope of the initial segment in the triangle above the segment from $(0,0)$ to $(1,0)$ has a slope m. We will show the following.

Theorem. *The s-line* l *is closed if* m *is a rational multiple of* $\sqrt{3}$, *and* l *is open if* m *is an irrational multiple of* $\sqrt{3}$.

In the case that m is a rational multiple of $\sqrt{3}$, we know that the slope of λ may change, but the slope will always be a rational multiple of $\sqrt{3}$. Lemmas 1 and 2 show that if λ passes through one vertex, then it must pass through infinitely many. If this is the case, and since the angle is constant modulo 30°, there must be infinitely many of these vertices where λ enters these vertices at precisely the same angle. Each of these corresponds to l entering a vertex on the s-manifold at a particular angle with one of the edges. Since this can only happen a finite number of ways, l must enter a particular vertex on the s-manifold an infinite number of times in exactly the same way. This can only happen if l is closed.

If λ does not pass through a vertex (and so l does not either), then λ is a straight line in the plane. Its initial point has coordinates $(c,0)$ with $0 < c < 1$ and slope $m = y\sqrt{3}/x$ where both x and y are integers. For each positive integer z, λ passes through the point $[c,0] + z(x-y)[1,0] + zy[1/2, \sqrt{3}/2]$. This is a point, which is a distance c from the left endpoint of the horizontal edge from some triangle in the tiling. This corresponds to l intersecting some edge in the s-manifold in a particular way, and this can only happen a finite number of different ways. It follows that l intersects a particular edge exactly the same way an infinite number of times, and this can only happen if l is closed.

On the other hand, if l is closed, and l passes through a vertex, then any projection must pass through infinitely many vertices. This can only happen if m is a rational multiple of $\sqrt{3}$. If l passes through no vertex, then its projection λ is a straight line. Since l is closed, it intersects edges in only finitely many different ways. Therefore, λ must intersect two horizontal edges in exactly the same way. In particular, for some $0 < c < 1$, the intersections must be $[c,0] + a[1,0] + b[1/2, \sqrt{3}/2]$ and $[c,0] + A[1,0] + B[1/2, \sqrt{3}/2]$. The slope is, therefore, $m = \sqrt{3}(B-b)/(2(A-a)+(B-b))$ a rational multiple of $\sqrt{3}$.

Conclusion

The axiom, each line is extendable to infinity, is S-denied in every closed s-manifold. We can say that this axiom is S-denied densely, since we can look at all of the s-lines at each point, and all the angles that correspond to closed s-lines are (topologically) dense in the interval $[0°,360°]$, as are the angles that correspond to open s-lines. In other words, within any angle emanating from a point P, no matter how small, there are closed and open s-lines in the interior of the angle.

The arguments presented here hold in any s-manifold, except for the parts depending on there being a finite number of triangles. In particular, if a projection of an s-line has a slope that is a rational multiple of $\sqrt{3}$, then it will intersect edges and vertices in only a limited number of ways. Its local structure, therefore, is in some sense periodic.

References

1. Iseri, H., *Smarandache Manifolds*, American Research Press, 2002.