A CONJECTURE CONCERNING INDEXES OF BEAUTY

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Abstract. In this paper we prove that 64 is not an index of beauty.

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For any positive integer n, let $d(n)$ be the number of distinct divisors of n. It is a well known fact that if

(1) $$n = p_1^{a_1} p_2^{a_2} \cdots p_k^{a_k}$$

is the factorization of n, then we have

(2) $$d(n) = (a_1 + 1)(a_2 + 1) \cdots (a_k + 1)$$

(see[1]). For a fixed positive integer m, if there exist a positive integer n such that

(3) $$m = \frac{n}{d(n)},$$

then m is called an index of beauty. Recently, Murthy [2] proposed the following conjecture:

Conjecture Every positive integer is an index of beauty.

In this paper we give a counter-example for the above-mentioned conjecture. We prove the following result:

Theorem 64 is not an index of beauty.

Proof We now suppose that 64 is an index of beauty. Then there exist a positive integer n such that

(4) $$n = 64d(n).$$

We see from (4) that n is even. Hence, n has the factorization

(5) $$n = 2^{a_0} p_1^{a_1} \cdots p_r^{a_r},$$

where $p_1, \cdots, p_r$ are odd primes with $p_1 < \cdots < p_r$, a_0 is a positive integer with $a_0 \geqslant 6$, $a_1, \cdots, a_r$ are positive integers. Let

(6) $$b = a_0 - 6.$$

By (4), (5) and (6), we get

(7) $$2^b p_1^{a_1} \cdots p_r^{a_r} = (b + 7)(a_1 + 1) \cdots (a_r + 1).$$

Since $p_1, \cdots, p_r$ are odd primes, we have

(8) $$p_i^{a_i} \geq \frac{2}{3}(a_i + 1), i = 1, \cdots, r.$$

From (7) and (8), we get

(9) $$b + 7 \geq 2^b \left(\frac{3}{2}\right)^r \geq 2^{b-1} 3.$$

It implies that $b \leqslant 2$.

If $b=2$, then from (7) we get $r=1$ and

(10) $$4 p_1^{a_1} = 9(a_1 + 1),$$

whence we get $p_1=3$, $a_1 \geqslant 2$ and

(11) $$4 \cdot 3^{a_1 - 2} = a_1 + 1.$$

Since $4 \cdot 3^{a_1-2} > 4(1 + (a_1 - 2)\log 3) > 4(a_1 - 1) > a_1 + 1$, (11) is impossible.

If $b=1$, then from (7) we get

(12) $$p_1^{a_1}\cdots p_r^{a_r}=4(a_1+1)\cdots(a_r+1).$$

Since $p_1, \cdots, p_r$ are odd primes, (12) is impossible.

If b=0, then from (7) we get

(13) $$p_1^{a_1}\cdots p_r^{a_r}=7(a_1+1)\cdots(a_r+1).$$

We see from (13) that $a_1+1, \cdots, a_r+1$ are odd. It implies that $a_1, \cdots, a_r$ are even. So we have $a_i \geqslant 2$ (i=1, $\cdots$, r) and

(14) $$p_i^{a_i}\geq 3(a_i+1), i=1,\cdots,r.$$

By (13) and (14), we get r=1. Further, by (13), we obtain p_1=7 and

(15) $$7^{a_1-1}=a_1+1.$$

However, since $a_1 \geqslant 2$, (15) is impossible. Thus, 64 is not an index of beauty. The theorem is proved.

References

[1] G. H. Hardy and E. M. Wright, An introduction to the theory of numbers, Oxford Univ. Press, Oxford, 1937.

[2] A. Murthy, Some more conjectures on primes and divisors, Smarandche Notions J. 12(2001), 311-312.