A functional recurrence to obtain the prime numbers using the Smarandache prime function.

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Theorem: We are considering the function:

For n ≥ 2 , integer:

$$F(n) = n + 1 + \sum_{m=n+1}^{2n} \prod_{i=n+1}^{m} \left[-E\left[-\frac{\sum_{j=1}^{i}\left(E\left(\frac{i}{j}\right)-E\left(\frac{i-1}{j}\right)\right)-2}{\sum_{j=1}^{i}\left(E\left(\frac{i}{j}\right)-E\left(\frac{i-1}{j}\right)\right)-1} \right] \right]$$

one has: p_{k+1} =F(p_k) for all $k \geq 1$ where $\{p_k\}_{k\geq 1}$ are the prime numbers and E(x) is the greatest integer less than or equal to x.

Observe that the knowledge of p_{k+1} only depends on knowledge of p_k and the knowledge of the fore primes is unnecessary.

Observe that this is a functional recurrence strictly closed too.

Proof:

Suppose that we have found a function $G(i)$ with the following property:

$$G(i) = \begin{cases} 1 & \text{if } i \text{ is compound} \\ 0 & \text{if } i \text{ is prime} \end{cases}$$

This function is called Smarandache Prime Function (Reference)

Consider the following product:

$$\prod_{i=p_k+1}^{m} G(i)$$

If $p_k < m < p_{k+1}$ $\prod_{i=p_k+1}^{m} G(i) = 1$ since $i : p_k + 1 \leq i \leq m$ are all compounds.

If $m \geq p_{k+1}$ $\prod_{i=p_k+1}^{m} G(i) = 0$ since the $G(p_{k+1}) = 0$ factor is in the product.

Here is the sum:

$$\sum_{m=p_k+1}^{2p_k} \prod_{i=p_k+1}^{m} G(i) = \sum_{m=p_k+1}^{p_{k+1}-1} \prod_{i=p_k+1}^{m} G(i) + \sum_{m=p_{k+1}}^{2p_k} \prod_{i=p_k+1}^{m} G(i) = \sum_{m=p_k+1}^{p_{k+1}-1} 1 =$$

$$= p_{k+1} - 1 - (p_k + 1) + 1 = p_{k+1} - p_k - 1$$

The second sum is zero since all products have the factor $G(p_{k+1}) = 0$.

Therefore we have the following relation of recurrence:

$$p_{k+1} = p_k + 1 + \sum_{m=p_k+1}^{2p_k} \prod_{i=p_k+1}^{m} G(i)$$

Let's now see that we can find $G(i)$ with the asked property.
Considerer:

$$(1) \quad E\left(\tfrac{i}{j}\right) - E\left(\tfrac{i-1}{j}\right) = \begin{cases} 1 & si\ j \mid i \\ 0 & si\ j \nmid i \end{cases} \qquad j = 1,2,\dots,i \qquad i \geq 1$$

We shall deduce this later.

We deduce of this relation:

$$d(i) = \sum_{j=1}^{i} \left(E\left(\tfrac{i}{j}\right) - E\left(\tfrac{i-1}{j}\right) \right) \text{ where } d(i) \text{ is the number of divisors of } i.$$

If i is prime $d(i) = 2$ therefore:

$$-E\left[-\tfrac{d(i)-2}{d(i)-1} \right] = 0$$

If i is compound $d(i) > 2$ therefore:

$$0 < \tfrac{d(i)-2}{d(i)-1} < 1 \Rightarrow -E\left[-\tfrac{d(i)-2}{d(i)-1} \right] = 1$$

Therefore we have obtained the function $G(i)$ which is:

$$G(i) = -E\left[-\frac{\sum_{j=1}^{i}\left(E\left(\frac{i}{j}\right)-E\left(\frac{i-1}{j}\right)\right)-2}{\sum_{j=1}^{i}\left(E\left(\frac{i}{j}\right)-E\left(\frac{i-1}{j}\right)\right)-1} \right] \qquad i \geq 2 \text{ integer}$$

To finish the demonstration of the theorem it is necessary to prove (1)

If $j = 1 \quad j \mid i \quad E\left(\frac{i}{j}\right) - E\left(\frac{i-1}{j}\right) = i - (i-1) = 1$

If $j > 1$

$$i = jE\left(\tfrac{i}{j}\right) + r \quad 0 \leq r < j$$
$$i - 1 = jE\left(\tfrac{i-1}{j}\right) + s \quad 0 \leq s < j$$

If $j \mid i \Rightarrow r = 0 \Rightarrow jE\left(\frac{i}{j}\right) = jE\left(\frac{i-1}{j}\right) + s + 1 \Rightarrow \left.\begin{array}{c} j \mid s+1 \\ s+1 \leq j \end{array}\right\} \Rightarrow j = s + 1$

$\Rightarrow jE\left(\frac{i}{j}\right) = jE\left(\frac{i-1}{j}\right) + j \Rightarrow E\left(\frac{i}{j}\right) = E\left(\frac{i-1}{j}\right) + 1$

If $j \nmid i \Rightarrow r > 0 \Rightarrow 0 = j\left(E\left(\frac{i}{j}\right) - E\left(\frac{i-1}{j}\right)\right) + (r - s) + 1 \Rightarrow j \mid r - s + 1$

Therefore $r - s + 1 = 0$ or $r - s + 1 = j$

If $s \neq 0 \Rightarrow r - s < j - 1 \Rightarrow r - s + 1 = 0 \Rightarrow E\left(\frac{i}{j}\right) = E\left(\frac{i-1}{j}\right)$

If $s = 0 \Rightarrow j \mid i - 1 \Rightarrow E\left(\frac{i}{j}\right) = E\left(\frac{i-1}{j} + \frac{1}{j}\right) = \frac{i-1}{j} = E\left(\frac{i-1}{j}\right)$

With this, the theorem is already proved.

Reference:

[1] E. Burton, "Smarandache Prime and Coprime Functions",
http://www.gallup.unm.edu/~smarandache/primfnct.txt

[2] F. Smarandache, "Collected Papers", Vol. II, 200 p., p. 137,
Kishinev University Press, Kishinev, 1997.