# A Generalisation of Euler's function

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The aim of this article is to propose a generalisation for Euler's function. This function is $\varphi : N \to N$ defined as follows $(\forall\, n \in N)\, \varphi(n) = \left|\{k = \overline{1,n} \mid (k,n) = 1\}\right|$. Perhaps, this is the most important function in number theory having many properties in number theory, combinatorics, *etc.* The main properties [Hardy & Wright, 1979] of this function are enumerated in the following:

$$(\forall\, a,b \in N)(a,b) = 1 \Rightarrow \varphi(a \cdot b) = \varphi(a) \cdot \varphi(b) \text{ - the multiplicative property} \quad (1)$$

$$a = p_1^{m_1} \cdot p_2^{m_2} \cdot \ldots \cdot p_s^{m_s} \Rightarrow \varphi(a) = a \cdot \left(1 - \tfrac{1}{p_1}\right) \cdot \left(1 - \tfrac{1}{p_2}\right) \cdot \ldots \cdot \left(1 - \tfrac{1}{p_s}\right) \quad (2)$$

$$(\forall\, a \in N) \sum_{d|a} \varphi(d) = a . \quad (3)$$

More properties concerning this function can be found in [Hardy & Wright, 1979], [Jones & Jones, 1998] or [Rosen, 1993].

## 1. Euler's Function by order *k*

In the following, we shall see how this function is generalised such that the above properties are still kept. The way that will be used to introduce Euler's generalised function is from the function's formula to the function's properties.

***Definition* 1.** Euler's function by order $k \in N$ is $\varphi_k : N \to N$ defined by

$$\left(\forall a = p_1^{m_1} \cdot p_2^{m_2} \cdot \ldots \cdot p_s^{m_s}\right) \varphi_k(a) = a^k \cdot \left(1 - \tfrac{1}{p_1^k}\right) \cdot \left(1 - \tfrac{1}{p_2^k}\right) \cdot \ldots \cdot \left(1 - \tfrac{1}{p_s^k}\right).$$

***Remarks* 1.**

1. Let us assume that $\varphi_k(1) = 1$.

2. Euler function by order 1 is Euler's function. Obviously, Euler's function by order 0 is the constant function 1.

In the following, the main properties of Euler's function by order $k$ are proposed.

***Theorem* 1.** Euler's function by order $k$ is multiplicative

$$(\forall\, a,b \in N)(a,b)=1 \Rightarrow \varphi_k(a\cdot b)=\varphi_k(a)\cdot\varphi_k(b). \tag{4}$$

**Proof**

This proof is obvious from the definition. ♣

***Theorem* 2.** $(\forall\, a \in N)\sum_{d|a}\varphi_k(d)=a^k$. (5)

**Proof**

The function $\overline{\varphi_k}(a)=\sum_{d|a}\varphi_k(d)$ is multiplicative because $\varphi_k$ is a multiplicative function.

If $a=p^m$, then the following transformation proves (5)

$$\overline{\varphi_k}(a)=\sum_{d|p^m}\varphi_k(d)=\sum_{i=0}^{m}\varphi_k(p^i)=1+\sum_{i=1}^{m}p^{k\cdot i}\left(1-\frac{1}{p^k}\right)=$$

$$=1+\sum_{i=1}^{m}\left(p^{k\cdot i}-p^{k\cdot(i-1)}\right)=1+p^{k\cdot m}-1=a^k$$

If $a=p_1^{m_1}\cdot p_2^{m_2}\cdot\ldots\cdot p_s^{m_s}$ then the multiplicative property is applied as follows:

$$\overline{\varphi_k}(a)=\overline{\varphi_k}(p_1^{m_1})\cdot\overline{\varphi_k}(p_2^{m_2})\cdot\ldots\cdot\overline{\varphi_k}(p_s^{m_s})=p_1^{k\cdot m_1}\cdot p_2^{k\cdot m_2}\cdot\ldots\cdot p_s^{k\cdot m_s}=a^k.$$ ♣

***Definition* 2.** A natural number $n$ is said to be $k$-power free if there is not a prim number $p$ such that $p^k \mid n$.

***Remarks* 2.**

1. There is not a 0-power free number.
2. Assume that 1 is the only 1-power free number.

The combinatorial property of Euler's function by order $k$ is given by the following theorem. This property is introduced by using the $k$-power free notion.

***Theorem* 3.** $(\forall\, n \in N)\, \varphi_k(n) = \left|\left\{i = \overline{1, n^k} \mid (i, n^k) \text{ is } k\text{ - power free}\right\}\right|$ (6)

**Proof**

This proof is made using the Inclusion-Exclusion theorem.

Let $a = p_1^{m_1} \cdot p_2^{m_2} \cdot \ldots \cdot p_s^{m_s}$ be the prime number decomposition of $a$.

If $d|n$, then the set $S_d = \left\{i = \overline{1, a^k} \mid d^k \mid i\right\}$ contains all the numbers that have the divisor $d^k$. This set satisfies the following properties:

$$S_d = \left\{d^k, 2 \cdot d^k, \ldots, \frac{a^k}{d^k} \cdot d^k\right\} \Rightarrow |S_d| = \frac{a^k}{d^k} \tag{7}$$

$$1 \le j_1 < j_2 \le n \Rightarrow S_{p_{j1}} \cap S_{p_{j2}} = S_{p_{j1} \cdot p_{j2}} \tag{8}$$

$$\left\{i = \overline{1, a^k} \mid (i, a^k) \text{ is } k\text{ - power free}\right\} = \left\{i = \overline{1, a^k}\right\} - \left(S_{p_1} \cap S_{p_2} \cap \ldots \cap S_{p_s}\right). \tag{9}$$

The Inclusion-Exclusion theorem and (7-9) give the following transformations:

$$\left|\left\{i = \overline{1, a^k} \mid (i, a^k) \text{ is } k\text{ - power free}\right\}\right| = a^k - \left|\left(S_{p_1} \cap S_{p_2} \cap \ldots \cap S_{p_s}\right)\right| =$$

$$= a^k - \sum_{j=1}^{s} \left|S_{p_j}\right| + \sum_{1 \le j_1 < j_2 \le n} \left|S_{p_{j1}} \cap S_{p_{j2}}\right| - \ldots + (-1)^{s+1} \sum_{1 \le j_1 < j_2 < \ldots < j_s \le n} \left|S_{p_{j1}} \cap S_{p_{j2}} \cap \ldots \cap S_{p_{js}}\right| =$$

$$= a^k - \sum_{j=1}^{s} \left|S_{p_j}\right| + \sum_{1 \le j_1 < j_2 \le n} \left|S_{p_{j1} \cdot p_{j2}}\right| - \ldots + (-1)^{s+1} \sum_{1 \le j_1 < j_2 < \ldots < j_s \le n} \left|S_{p_{j1} \cdot p_{j2} \cdot \ldots \cdot p_{j2}}\right| =$$

$$= a^k - \sum_{j=1}^{s} \frac{a^k}{p_j^k} + \sum_{1 \le j_1 < j_2 \le n} \frac{a^k}{\left(p_{j_1} \cdot p_{j_2}\right)^k} - \ldots + (-1)^{s+1} \sum_{1 \le j_1 < j_2 < \ldots < j_s \le n} \frac{a^k}{\left(p_{j_1} \cdot p_{j_2} \cdot \ldots \cdot p_{j_2}\right)^k} =$$

$$= a^k \cdot \left(1 - \frac{1}{p_1^k}\right) \cdot \left(1 - \frac{1}{p_2^k}\right) \cdot \ldots \cdot \left(1 - \frac{1}{p_s^k}\right) = \varphi_k(a)$$

Therefore, the equation (6) holds. ♣

## 3. Conclusion

Euler's function by order $k$ represents a successful way to generalise Euler's function. Firstly, because the main properties of Euler's function (1-3) have been extended for Euler's function by order $k$. Secondly and more important, because a combinatorial property has been found for this generalised function. Obviously, many other properties can be deduced for Euler's function by order $k$. Unfortunately, a similar property with Euler's theorem $a^{\varphi(n)} = 1 \bmod n$ has not been found so far.

## References

Hardy, G. H. and Wright, E. M. (1979) *Introduction to the Theory of Numbers*, (5th ed.) Oxford University Press, Oxford.

Jones, G. A. and Jones, M. J. (1998) *Elementary Number Theory*, Springer-Verlag, New York.

Rosen, K (1993) *Elementary Number Theory and its Application*, Addison-Wesley, New York.