A LINEAR COMBINATION WITH SMARANDACHE FUNCTION TO OBTAIN THE IDENTITY[1]

by
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In this paper we consider a numerical function $i_p : N^* \to N$ (p is an arbitrary prime number) associated with a particular Smarandache Function $S_p : N^* \to N$ such that $(1/p)S_p(a) + i_p(a) = a$.

1. INTRODUCTION. In [7] is defined a numerical function $S : N^* \to N$, $S(n)$ is the smallest integer such that $S(n)!$ is divisible by n. This function may be extended to all integers by defining $S(-n) = S(n)$.

If a and b are relatively prime then $S(a \cdot b) = \max\{S(a), S(b)\}$, and if $[a, b]$ is the last common multiple of a and b then $S([a \cdot b]) = \max\{S(a), S(b)\}$.

Suppose that $n = p_1^{a_1} p_2^{a_2} \ldots . p_r^{a_r}$ is the factorization of n into primes. In this case,

$$S(n) = \max\{S(p_i^{a_i} | i = 1, \ldots, r\} \tag{1}$$

Let $a_n(p) = (p^n - 1)/(p-1)$ and $[p]$ be the generalized numerical scale generated by $(a_n(p))_{n \in N}$:

$$[p] : a_1(p), a_2(p), \ldots ., a_n(p), \ldots$$

By (p) we shall note the standard scale induced by the net $b_n(p) = p^n$:

$$(p) : 1, p, p^2, p^3, \ldots, p^n, \ldots$$

In [2] it is proved that

$$S(p^a) = p(a_{[p]})_{[p]} \tag{2}$$

That is the value of $S(p^a)$ is obtained multiplying by p the number obtained writing the exponent a in the generalized scale [p] and "reading" it in the standard scale (p).

Let us observe that the calculus in the generalized scale [p] is different from the calculus in the standard scale (p), because

$$a_{n+1}(p) = pa_n(p) + 1 \text{ and } b_{n+1}(p) = pb_n(p) \tag{3}$$

We have also

$$a_m(p) \le a \Leftrightarrow (p^m - 1)/(p-1) \le a \Leftrightarrow p^m \le (p-1) \cdot a + 1 \Leftrightarrow m \le \log_p((p-1) \cdot a + 1)$$

so if

$$a_{[p]} = v_t a_t(p) + v_{t-1} a_{t-1}(p) + \ldots + v_1 a_1(p) = \overline{v_t v_{t-1} \ldots v_{1[p]}}$$

is the expression of a in the scale [p] then t is the integer part of $\log_p((p-1) \cdot a + 1)$

$$t = [\log_p((p-1) \cdot a + 1)]$$

and the digit v_t is obtained from $a = v_t a_t(p) + r_{t-1}$.

In [1] it is proved that

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$$S(p^a) = (p-1)\cdot a + \sigma_{[p]}(a) \tag{4}$$

where $\sigma_{[p]}(a) = v_1 + v_2 + ... + v_\upsilon$.

A Legendre formula asert that

$$a! = \prod_{\substack{p_i \le a \\ p_i \text{ prim}}} p_i^{E_{p_i}(a)}$$

where $E_p(a) = \sum_{j\ge1} \left[\frac{a}{p^j}\right]$.

We have also that ([5])

$$E_p(a) = \frac{\left(a - \sigma_{[p]}(a)\right)}{p-1} \tag{5}$$

and ([1]) $E_p(a) = \left(\left[\frac{a}{p}\right]_{(p)}\right)_{[p]}$.

In [1] is given also the following relation between the function E_p and the Smarandache function

$$S(p^a) = \frac{(p-1)^2}{p}\left(E_p(a) + a\right) + \frac{p-1}{p}\sigma_{[p]}(a) + \sigma_{[p]}(a)$$

There exist a great number of problems concerning the Smarandache function. We present some of these problem.

P. Gronas find ([3]) the solution of the diophantine equation $F_S(n) = n$, where $F_S(n) = \sum_{d|n} S(d)$. The solution are n=9, n=16 or n=24, or n=2p, where p is a prime number.

T. Yau ([8]) find the triplets which verifies the Fibonacci relationship

$$S(n) = S(n+1) + S(n+2).$$

Checking the first 1200 numbers, he find just two triplets which verifies this relationship: (9,10,11) and (119,120,121). He can't find theoretical proof.

The following conjecture that: "the equation $S(x) = S(x+1)$, has no solution", was not completely solved until now.

2. The Function $i_p(a)$. In this section we shall note $S(p^a) = S_p(a)$. From the Legendre formula it results ([4]) that

$$S_p(a) = p\left(a - i_p(a)\right) \text{ with } 0 \le i_p(a) \le \left[\frac{a-1}{p}\right]. \tag{6}$$

That is we have

$$\frac{1}{p}S_p(a) + i_p(a) = a \tag{7}$$

and so for each function S_p there exists a function i_p such that we have the linear combination (7) to obtain the identity.

In the following we keep out some formulae for the calculus of i_p. We shall obtain a duality relation between i_p and E_p.

Let $a_{(p)} = \overline{u_k u_{k-1} .....u_1 u_0} = u_k p^k + u_{k-1} p^{k-1} + ... + u_1 p + u_0$.

Then

$$a=(p-1)\left(u_k\frac{p^k-1}{p-1}+u_{k-1}\frac{p^{k-1}-1}{p-1}+\ldots+u_1\frac{p-1}{p-1}\right)+(u_k+u_{k-1}+\ldots+u_1)+u_0=$$

$$(p-1)\left(\left[\frac{a}{p}\right]_{(p)}\right)_{[p]}+\sigma_{(p)}(a)=(p-1)E_{p(a)}+\sigma_{(p)}(a) \tag{8}$$

From (4) it results

$$a=\frac{S_p(a)-\sigma_{[p]}(a)}{p-1} \tag{9}$$

From (8) and (9) we deduce

$$(p-1)E_p(a)+\sigma_{(p)}(a)=\frac{S_p(a)-\sigma_{[p]}(a)}{p-1}.$$

So,

$$S_p(a)=(p-1)^2E_p(a)+(p-1)\sigma_{(p)}(a)+\sigma_{[p]}(a) \tag{10}$$

From (4) and (7) it results

$$i_p(a)=\frac{a-\sigma_{[p]}(a)}{p} \tag{11}$$

and it is easy to observe a complementary with the equality (5).
Combining (5) and (11) it results

$$i_p(a)=\frac{(p-1)E_p(a)+\sigma_{(p)}(a)-\sigma_{[p]}}{p} \tag{12}$$

From

$$a=\overline{\upsilon_t\upsilon_{t-1}\ldots\upsilon_1}_{[p]}=\upsilon_t(p^{t-1}+p^{t-2}+\ldots\ldots+p+1)+\upsilon_{t-1}(p^{t-2}+p^{t-3}+\ldots\ldots+p+1)+$$
$$+\ldots+\upsilon_2(p+1)+\upsilon_1$$

it results that

$$a=(\upsilon_tp^{t-1}+\upsilon_{t-1}p^{t-2}+\ldots+\upsilon_2p+\upsilon_1)+\upsilon_t(p^{t-2}+p^{t-1}+\ldots+1)+\upsilon_{t-1}(p^{t-3}+p^{t-4}+\ldots+1)+\ldots+$$
$$\upsilon_3(p+1)+\upsilon_2=(a_{[p]})_{(p)}+\left[\frac{a}{p}\right]-\left[\frac{\sigma_{[p]}(a)}{p}\right]$$

because

$$\left[\frac{a}{p}\right]=\left[\upsilon_t(p^{t-2}+p^{t-3}+\ldots+p+1)+\frac{\upsilon_t}{p}+\upsilon_{t-1}(p^{t-3}+p^{t-4}+\ldots+p+1)+\frac{\upsilon_{t-1}}{p}+\ldots+\right.$$
$$\left.+\upsilon_3(p+1)+\frac{\upsilon_3}{p}+\upsilon_2+\frac{\upsilon_2}{p}+\frac{\upsilon_1}{p}\right]=\upsilon_t(p^{t-2}+p^{t-3}+\ldots+p+1)+$$
$$+\upsilon_{t-1}(p^{t-3}+p^{t-4}+\ldots+p+1)+\ldots+\upsilon_3(p+1)+\upsilon_2+\left[\frac{\sigma_{[p]}(a)}{p}\right]$$

we have $[n+x]=n+[x]$.
Then

$$a=(a_{[p]})_{(p)}+\left[\frac{a}{p}\right]-\left[\frac{\sigma_{[p]}(a)}{p}\right] \tag{13}$$

or

$$a=\frac{S_p(a)}{p}+\left[\frac{a}{p}\right]-\left[\frac{\sigma_{[p]}(a)}{p}\right]$$

It results that

$$S_p(a)=p\left(a-\left(\left[\frac{a}{p}\right]-\left[\frac{\sigma_{[p]}(a)}{p}\right]\right)\right) \tag{14}$$

From (11) and (14) we obtain

$$i_p(a)=\left[\frac{a}{p}\right]-\left[\frac{\sigma_{[p]}(a)}{p}\right] \tag{15}$$

It is know that there exists $m,n\in N$ such that the relation

$$\left[\frac{m-n}{p}\right]=\left[\frac{m}{p}\right]-\left[\frac{n}{p}\right] \tag{16}$$

is not verifies.

But if $\frac{m-n}{p}\in N$ then the relation (16) is satisfied.

From (11) and (15) it results

$$\left[\frac{a-\sigma_{[p]}(a)}{p}\right]=\left[\frac{a}{p}\right]-\left[\frac{\sigma_{[p]}(a)}{p}\right].$$

This equality results also by the fact that $i_p(a)\in N$.

From (2) and (11) or from (13) and (15) it results that

$$i_p(a)=a-\left(a_{[p]}\right)_{(p)} \tag{17}$$

From the condition on i_p in (6) it results that $\Delta=\left[\frac{a-1}{p}\right]-i_p(a)\geq 0$.

To calculate the difference $\Delta=\left[\frac{a-1}{p}\right]-i_p(a)$ we observe that

$$\Delta=\left[\frac{a-1}{p}\right]-i_p(a)=\left[\frac{a-1}{p}\right]-\left[\frac{a}{p}\right]+\left[\frac{\sigma_{[p]}(a)}{p}\right] \tag{18}$$

For $a\in[kp+1,kp+p-1]$ we have $\left[\frac{a-1}{p}\right]=\left[\frac{a}{p}\right]$ so

$$\Delta=\left[\frac{a-1}{p}\right]-i_p(a)=\left[\frac{\sigma_{[p]}(a)}{p}\right] \tag{19}$$

If $a=kp$ then $\left[\frac{a-1}{p}\right]=\left[\frac{kp-1}{p}\right]=\left[k-\frac{1}{p}\right]=k-1$ and $\left[\frac{a}{p}\right]=k$.

So, (18) becomes

$$\Delta=\left[\frac{a-1}{p}\right]-i_p(a)=\left[\frac{\sigma_{[p]}(a)}{p}\right]-1 \tag{20}$$

Analogously, if $a=kp+p$, we have

$$\left[\frac{a-1}{p}\right]=\left[\frac{p(k+1)-1}{p}\right]=\left[k+1-\frac{1}{p}\right]=k \quad \text{and} \quad \left[\frac{a}{p}\right]=k+1$$

so, (18) has the form (20).

For any number a, for which Δ is given by (19) or by (20), we deduce that Δ is maximum when $\sigma_{[p]}(a)$ is maximum, so when

$$a_M=\overline{\underbrace{(p-1)(p-1)...(p-1)}_{t\ \text{terms}}p}_{[p]} \tag{21}$$

That is

$$a_M=(p-1)a_t(p)+(p-1)a_{t-1}(p)+....+(p-1)a_2(p)+p=$$
$$=(p-1)\left(\frac{p^t-1}{p-1}+\frac{p^{t-1}-1}{p-1}+...+\frac{p^2-1}{p-1}\right)+p=$$
$$=(p^t+p^{t-1}+...+p^2+p)-(t-1)=pa_t(p)-(t-1)$$

It results that a_M is not multiple of p if and only if $t-1$ is not a multiple of p.

In this case $\sigma_{[p]}(a)=(t-1)(p-1)+p=pt-t+1$ and

$$\Delta=\left[\frac{\sigma_{[p]}(a)}{p}\right]=\left[t-\frac{t-1}{p}\right]=t-\left[\frac{t-1}{p}\right].$$

So $i_p(a_M)\geq\left[\frac{a_M-1}{p}\right]-t$ or $i_p(a_M)\in\left[\left[\frac{a_M-1}{p}\right]-t,\left[\frac{a_M-1}{p}\right]\right]$. If $t-1\in(kp,kp+p)$ then $\left[\frac{t-1}{p}\right]=k$ and $k(p-1)+1<\Delta(a_M)<k(p-1)+p+1$ so $\lim_{a_M\to\infty}\Delta(a_M)=\infty$.

We also observe that

$$\left[\frac{a_M-1}{p}\right]=a_t(p)-\left[\frac{t-1}{p}\right]=\frac{p^{t+1}-1}{p-1}-\left[\frac{t-1}{p}\right]\in\left[\frac{p^{kp+1}-1}{p-1}-k,\frac{p^{kp+p+1}-1}{p-1}-k\right].$$

Then if $a_M\to\infty$ (as p^x), it results that $\Delta(a_M)\to\infty$ (as x).

From $\frac{i_p(a_M)}{\left[\frac{a_M-1}{p}\right]}=\frac{a_t(p)-t}{a_t(p)-\left[\frac{t-2}{p}\right]}\to 1$ it results $\lim_{a\to\infty}\frac{i_p(a)}{[a-1]p}=1$.

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