A NEW SEQUENCE RELATED SMARANDACHE SEQUENCES AND ITS MEAN VALUE FORMULA*

ZHANG WENPENG

Research Center for Basic Science, Xi'an Jiaotong University
Xi'an, Shaanxi, P.R.China

ABSTRACT. Let n be any positive integer, $a(n)$ denotes the product of all non-zero digits in base 10. For natural $x \geq 2$ and arbitrary fixed exponent $m \in N$, let $A_m(x) = \sum\limits_{n<x} a^m(n)$. The main purpose of this paper is to give two exact calculating formulas for $A_1(x)$ and $A_2(x)$.

1. INTRODUCTION

For any positive integer n, let $b(n)$ denotes the product of base 10 digits of n. For example, $b(1) = 1$, b(2)=2,$\cdots$, $b(10) = 0$, $b(11) = 1$, $\cdots\cdot$. In problem 22 of book [1], Professor F.Smaradache ask us to study the properties of sequence $\{b(n)\}$. About this problem, it appears that no one had studied it yet, at least, we have not seen such a paper before. The problem is interesting because it can help us to find some new distribution properties of the base 10 digits. In this paper, we consider another sequence $\{a(n)\}$, which related to Smarandache sequences. Let $a(n)$ denotes the product of all non-zero digits in base 10 of n. For example, $a(1) = 1$, $a(2) = 2$, $a(12) = 2$, $\cdots$, $a(28) = 16$, $a(1023) = 6, \cdots\cdots$. For natural number $x \geq 2$ and arbitrary fixed exponent $m \in N$, let

$$A_m(x) = \sum_{n<x} a^m(n). \tag{1}$$

The main purpose of this paper is to study the calculating problem of $A_m(x)$, and use elementary methods to deduce two exact calculating formulas for $A_1(x)$ and $A_2(x)$. That is, we shall prove the following:

Theorem. *For any positive integer x, let $x = a_1 10^{k_1} + a_2 10^{k_2} + \cdots + a_s 10^{k_s}$ with $k_1 > k_2 > \cdots > k_s \geq 0$ and $1 \leq a_i \leq 9$, $i = 2, 3, \cdots, s$. Then we have the calculating formulas*

$$A_1(x) = \frac{a_1 a_2 \cdots a_s}{2} \sum_{i=1}^{s} \frac{a_i^2 - a_i + 2}{\prod\limits_{j=i}^{s} a_j} \left(45 + \left[\frac{1}{k_i+1}\right]\right) \cdot 46^{k_i - 1};$$

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$$A_2(x) = \frac{a_1^2 a_2^2 \cdots a_s^2}{6} \sum_{i=1}^{s} \frac{2a_i^3 - 3a_i^2 + a_i + 6}{\prod_{j=i}^{s} a_j^2} \left(285 + \left[\frac{1}{k_i+1}\right]\right) \cdot 286^{k_i-1},$$

where $[x]$ denotes the greatest integer not exceeding x.

For general integer $m \geq 3$, using our methods we can also give an exact calculating formula for $A_m(x)$. That is, we have the calculating formula

$$A_m(x) = a_1^m a_2^m \cdots a_s^m \sum_{i=1}^{s} \frac{1 + B_m(a_i)}{\prod_{j=i}^{s} a_j^m} \left(\left[\frac{1}{k_i+1}\right] + B_m(10)\right) \cdot (1 + B_m(10))^{k_i-1},$$

where a_i as the definition as in the above Theorem, and $B_m(N) = \sum_{1 \leq n < N} n^m$.

2. Proof of the Theorem

In this section, we complete the proof of the Theorem. First we need following two simple Lemmas.

Lemma 1. *For any integer $k \geq 1$ and $1 \leq c \leq 9$, we have the identities*

a) $A_1(10^k) = 45 \cdot 46^{k-1}$;

b) $A_1(c \cdot 10^k) = 45 \cdot \left(1 + \frac{(c-1)c}{2}\right) \cdot 46^{k-1}$.

Proof. We first prove a) of Lemma 1 by induction. For $k = 1$, we have $A_1(10^1) = A_1(10) = 1 + 2 + \cdots + 9 = 45$. So that the identity

$$A_1(10^k) = \sum_{n<10^k} a(n) = 45 \cdot 46^{k-1} \tag{2}$$

holds for $k = 1$. Assume (2) is true for $k = m \geq 1$. Then by the inductive assumption we have

$$\begin{aligned}
A_1(10^{m+1}) &= \sum_{n<9\cdot 10^m} a(n) + \sum_{9\cdot 10^m \leq n < 10^{m+1}} a(n) \\
&= A_1(9 \cdot 10^m) + \sum_{0 \leq n < 10^m} a(n + 9 \cdot 10^m) \\
&= A_1(9 \cdot 10^m) + 9 \cdot \sum_{0 \leq n < 10^m} a(n) \\
&= A_1(9 \cdot 10^m) + 9 \cdot \sum_{n < 10^m} a(n) \\
&= A_1(9 \cdot 10^m) + 9 \cdot A_1(10^m) \\
&= A_1(8 \cdot 10^m) + 9 \cdot A_1(10^m) + 8 \cdot A_1(10^m) \\
&= \cdots\cdots\cdots \\
&= (1 + 1 + 2 + 3 + 4 + 5 + 6 + 7 + 8 + 9) \cdot A_1(10^m) \\
&= 46 \cdot A_1(10^m) \\
&= 45 \cdot 46^m.
\end{aligned}$$

That is, (2) is true for $k = m+1$. This proves the first part of Lemma 1.

The second part b) follows from a) of Lemma 1 and the recurrence formula

$$\begin{aligned} A_1(c\cdot 10^k) &= \sum_{n<(c-1)\cdot 10^k} a(n) + \sum_{(c-1)\cdot 10^k \le n < c\cdot 10^k} a(n) \\ &= \sum_{n<(c-1)\cdot 10^k} a(n) + \sum_{0\le n<10^k} a(n+(c-1)\cdot 10^k) \\ &= \sum_{n<(c-1)\cdot 10^k} a(n) + (c-1)\cdot \sum_{n<10^k} a(n) \\ &= A_1((c-1)\cdot 10^k) + (c-1)\cdot A_1(10^k). \end{aligned}$$

This completes the proof of Lemma 1.

Lemma 2. *For any integer $k \ge 1$ and $1 \le c \le 9$, we have the identities*

c) $A_2(10^k) = 285 \cdot 286^{k-1}$;

d) $A_2(a\cdot 10^k) = 285\cdot \left[1 + \dfrac{(a-1)a(2a-1)}{6}\right]\cdot 286^{k-1}$.

Proof. Note that $A_2(10) = 285$. The Lemma 2 can be deduced by Lemma 1, induction and the recurrence formula

$$\begin{aligned} A_2(10^{k+1}) &= \sum_{n<9\cdot 10^k} a^2(n) + \sum_{9\cdot 10^k \le n < 10^{k+1}} a^2(n) \\ &= \sum_{n<9\cdot 10^k} a^2(n) + \sum_{0\le n<10^k} a^2(n+9\cdot 10^k) \\ &= \sum_{n<9\cdot 10^k} a^2(n) + 9^2\cdot \sum_{0\le n<10^k} a^2(n) \\ &= A_2(9\cdot 10^k) + 9^2\cdot A_2(10^k) \\ &= \cdots\cdots\cdots \\ &= (1+1^2+2^2+\cdots+9^2)\cdot A_2(10^k) \\ &= 286\cdot A_2(10^k). \end{aligned}$$

This completes the proof of Lemma 2.

Now we use Lemma 1 and Lemma 2 to complete the proof of the Theorem. For any positive integer x, let $x = a_1\cdot 10^{k_1} + a_2\cdot 10^{k_2} + \cdots + a_s\cdot 10^{k_s}$ with $k_1 > k_2 > \cdots > k_s \ge 0$ under the base 10. Then applying Lemma 1 repeatedly we have

$$\begin{aligned} A_1(x) &= \sum_{n<a_1\cdot 10^{k_1}} a(n) + \sum_{a_1\cdot 10^{k_1}\le n<x} a(n) \\ &= A_1(a_1\cdot 10^{k_1}) + \sum_{0\le n<x-a_1\cdot 10^{k_1}} a(n+a_1\cdot 10^{k_1}) \\ &= A_1(a_1\cdot 10^{k_1}) + a_1\cdot \sum_{0\le n<x-a_1\cdot 10^{k_1}} a(n) \end{aligned}$$

$$= A_1(a_1 \cdot 10^{k_1}) + a_1 \cdot A_1(x - a_1 \cdot 10^{k_1})$$
$$= A_1(a_1 \cdot 10^{k_1}) + a_1 \cdot A_1(a_2 \cdot 10^{k_2}) + a_1 a_2 \cdot A_1(x - a_1 \cdot 10^{k_1} - a_2 \cdot 10^{k_2})$$
$$= \cdots\cdots\cdots$$
$$= \sum_{i=1}^{s} \frac{a_1 a_2 \cdots a_s}{a_i a_{i+1} \cdots a_s} A_1(a_i \cdot 10^{k_i})$$
$$= a_1 a_2 \cdots a_s \sum_{i=1}^{s} \frac{(1 + \frac{(a_i-1)a_i}{2})}{\prod_{j=i}^{s} a_j} \left(45 + \left[\frac{1}{k_i+1}\right]\right) 46^{k_i-1}.$$

This proves the first part of the Theorem.

Applying Lemma 2 and the first part of the Theorem repeatedly we have

$$A_2(x) = \sum_{n < a_1 \cdot 10^{k_1}} a^2(n) + \sum_{a_1 \cdot 10^{k_1} \leq n < x} a^2(n)$$
$$= A_2(a_1 \cdot 10^{k_1}) + \sum_{0 \leq n < x - a_1 \cdot 10^{k_1}} a^2(n + a_1 \cdot 10^{k_1})$$
$$= A_2(a_1 \cdot 10^{k_1}) + a_1^2 \cdot \sum_{0 \leq n < x - a_1 \cdot 10^{k_1}} a^2(n)$$
$$= A_2(a_1 : 10^{k_1}) + a_1^2 \cdot A_2(x - a_1 \cdot 10^{k_1})$$
$$= A_2(a_1 \cdot 10^{k_1}) + a_1^2 \cdot A_2(a_2 \cdot 10^{k_2}) + a_1^2 a_2^2 \cdot A_2(x - a_1 \cdot 10^{k_1} - a_2 \cdot 10^{k_2})$$
$$= \cdots\cdots\cdots$$
$$= \sum_{i=1}^{s} \frac{a_1^2 a_2^2 \cdots a_s^2}{\prod_{j=i}^{s} a_j^2} A_2(a_i \cdot 10^{k_i})$$
$$= \frac{a_1^2 a_2^2 \cdots a_s^2}{6} \sum_{i=1}^{s} \frac{2a_i^3 - 3a_i^2 + a_i + 6}{\prod_{j=i}^{s} a_j^2} \left(285 + \left[\frac{1}{k_i+1}\right]\right) \cdot 286^{k_i-1}.$$

This completes the proof of the second part of the Theorem.

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