A NOTE ON S(n), WHERE n IS AN EVEN PERFECT NUMBER

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In a recent paper [1] the following result is proved: If $n = 2^{k-1}(2^k-1)$, 2^k-1 = prime, is an even perfect number, then $S(n) = 2^k-1$, where $S(n)$ is the well-known Smarandache Function.

Since $S(ab) = \max\{S(a), S(b)\}$ for $(a, b) = 1$, and $S(a) \le a$ with equality for $a = 1, 4$, and a = prime (see [3]), we have the folowwing one-line proof of this result:
$S(\ 2^{k-1}(2^k-1)\ ) = \max\{\ S(2^{k-1}),\ S(2^k-1)\ \} = 2^k-1$,
since $S(2^{k-1}) \le 2^{k-1} < 2^k-1$ for $k \ge 2$.
On the other hand, if 2^k-1 is prime, then we have $S(2^k-1) \equiv 1 \pmod k$; an interesting table is considered in [2]. Indeed, k must be a prime too, $k = p$; while Fermat's little theorem gives $2^p-1 \equiv 1 \pmod p$. From $2^{2p}-1 = (2^p-1)(2^p+1)$ and $(2^p-1, 2^p+1) = 1$ we can deduce $S(2^{2p}-1) = \max\{\ S(2^p-1),\ S(2^p+1)\ \} = 2^p-1$ since 2^p+1 is being composite, $S(2^p+1) < 2/3(2^p+1) < 2^p-1$ for $p \ge 3$. Thus, if 2^k-1 is a Mersenne prime, then $S(2^k-1) \equiv S(2^{2k}-1) \equiv 1 \pmod k$. If 2^p-1 and $2^{2p}+1$ are both primes, then
$S(2^{4p}-1) = \max\{\ S(2^{2p}-1),\ S(2^{2p}+1)\ \} = 2^{2p}+1 \not\equiv 1 \pmod{4p}$.

References:

1. Sebastian Martin Ruiz, "Smarandache's Function Applied to Perfect Numbers", <Smarandache Notions J.>, 10 (1999), No. 1-2-3, 114-115.
2. Sebastian Martin Ruiz, "A Congruence with Smarandache's Function", <Smarandache Notions J.>, 10 (1999), No. 1-2-3, 130-132.
3. Charles Ashbacher, "An Introduction to the Smarandache Function", Erhus Univ. Press, Vail, AZ, 1995.