A Note On $S(n^2)$

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In the paper[1], it is shown that

$$S(n^2) \leq n \text{ for } n > 4 \text{ and even.} \qquad (1)$$

In this note, we will prove that (1) holds for all $n > 4$, $n \neq$ prime.

Let p be a prime. Then:

Lemma: For $n \neq p$, $2p$, 8 or 9, we have

$$n^2 \mid (n-1)! \qquad (2)$$

Proof: If $n \neq p, 2p, p^2$, 8, or 16, then n can be written as $n = xy$ ($x \neq y$; $x, y \geq 3$). If $n \neq 16$, then $n = xy$ with $x \geq 3$, $y \geq 5$. Let $n = xy$ with $y > x$; $x \geq 3$. Now $x, y, 2x, 2y, 3x < n-1$; x, y, 2y are different and one of 2x, 3x is different from x, y, 2y. Therefore, $(n-1)!$ contains x, y, 2y and 2x or x, y, 2y and 3x. In any case one has $(n-1)! \mid x^2y^2 = n^2$.

If $n = p^2$, then $n - 1 > 4p$, thus $(n-1)!$ contains the factors p, 2p, 3p, 4p, so $(n-1)! \mid p^4 = n^2$. For $n = 2p$, clearly p^2 does not divide $(n-1)!$. For $n = 8$ or 9, n^2 does not divide $(n-1)!$, but for $n = 16$, this holds true by a simple verification.

As a corollary of (2), we can write

$$S(n^2) \leq n - 1 \text{ for } n \neq p, 2p, 8 \text{ or } 9 \qquad (3)$$

Since 2p and 8 are even and $S(9^2) = 9$, on the basis of (3), (1) holds true for $n \neq p$, $n > 4$.

Reference:

1. J. Sàndor, *On Certain New Inequalities and Limits for the Smarandache Function*, **SNJ** 9(1998), No. 1-2, 63-69.