A NOTE ON THE 29-TH SMARANDACHE'S PROBLEM*

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ABSTRACT. Let n be a positive integer, $a_k(n)$ be the k-th complement number of n. In this paper, we study the mean value properties of the k-th complement number sequences, and give an interesting asymptotic formula.
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1. INTRODUCTION

For any positive integer n to find the smallest integer $a_k(n)$ such that $na_k(n)$ is a perfect k-power($k \geq 2$), we define that $a_k(n)$ is the k-th complement number of n. Let $n = p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_m^{\alpha_m}$, then $a_k(n) = p_1^{\beta_1} p_2^{\beta_2} \cdots p_m^{\beta_m}$, where $\alpha_i + \beta_i \equiv 0(\bmod k)$ and $\beta_i < k$, $i = 1, 2, \cdots, m$. In problem 29 of [1], Professor F.Smarandach asked us to study the properties of the k-th complement number sequences. In this paper, we use the analytic methods to study the mean value properties of this sequences, and give an interesting asymptotic formula. That is, we shall prove the following:

Theorem. *For any positive number $x > 1$, we have the asymptotic formula*

$$\sum_{n \leq x} \frac{d(a_k(n))}{\phi(a_k(n))} = kx^{\frac{1}{k}} g(k) + O\left(x^{\frac{1}{2k}+\varepsilon}\right),$$

where $g(k) = \prod_p \left[1 + \frac{k}{p^{\frac{1}{k}+k-2}(p-1)} + \frac{k-1}{p^{\frac{2}{k}+k-3}(p-1)} + \cdots + \frac{2}{p^{\frac{k-1}{k}}(p-1)}\right]$, $d(n)$ *is the Dirichlet divisor function, $\phi(n)$ is Euler function, ε is any fixed positive number.*

Especially taking $k = 2$, we have

Corollary. *For any positive number $x > 1$, we have the asymptotic formula*

$$\sum_{n \leq x} \frac{d(a_2(n))}{\phi(a_2(n))} = 2x^{\frac{1}{2}} \prod_p \left(1 + \frac{2}{\sqrt{p}(p-1)}\right) + O\left(x^{\frac{1}{4}+\varepsilon}\right).$$

Key words and phrases. complement number; mean value properties; asymptotic formula.
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2. Proof of the Theorem

In this section, we shall complete the proof of the Theorem. Let

$$f(s) = \sum_{n=1}^{\infty} \frac{d(a_k(n))}{\phi(a_k(n))n^s}.$$

From the Euler product formula [2] and the definition of $a_k(n)$ we have

$$\begin{aligned}
f(s) &= \sum_{n=1}^{\infty} \frac{d(a_k(n))}{\phi(a_k(n))n^s} \\
&= \prod_p \left(1 + \frac{d(a_k(p))}{\phi(a_k(p))p^s} + \frac{d(a_k(p^2))}{\phi(a_k(p^2))p^{2s}} + \cdots + \frac{d(a_k(p^k))}{\phi(a_k(p^k))p^{ks}} + \frac{d(a_k(p^{k+1}))}{\phi(a_k(p^{k+1}))p^{(k+1)s}} + \cdots\right) \\
&= \prod_p \left(1 + \frac{d(p^{k-1})}{\phi(p^{k-1})p^s} + \frac{d(p^{k-2})}{\phi(p^{k-2})p^{2s}} + \cdots + \frac{1}{p^{ks}} + \frac{d(p^{k-1})}{\phi(p^{k-1})p^{(k+1)s}} + \cdots\right) \\
&= \prod_p \left[\frac{1}{1-\frac{1}{p^{ks}}} + \frac{d(p^{k-1})}{\phi(p^{k-1})p^s\left(1-\frac{1}{p^{ks}}\right)} + \cdots + \frac{d(p)}{\phi(p)p^{(k-1)s}\left(1-\frac{1}{p^{ks}}\right)}\right] \\
&= \zeta(ks) \prod_p \left[1 + \frac{d(p^{k-1})}{\phi(p^{k-1})p^s} + \frac{d(p^{k-2})}{\phi(p^{k-2})p^{2s}} + \cdots + \frac{d(p)}{\phi(p)p^{(k-1)s}}\right] \\
&= \zeta(ks) \prod_p \left[1 + \frac{k}{p^{k-2}(p-1)p^s} + \frac{k-1}{p^{k-3}(p-1)p^{2s}} + \cdots + \frac{2}{(p-1)p^{(k-1)s}}\right].
\end{aligned}$$

where $\zeta(s)$ is Riemann-zeta function. Taking $b = \frac{1}{k} + \frac{1}{logx}$, $T = x^{\frac{1}{2k}}$, then by Perron formula [3] we have

$$\begin{aligned}
\sum_{n \le x} \frac{d(a_k(n))}{\phi(a_k(n))} &= \frac{1}{2\pi i}\int_{b-iT}^{b+iT} f(s)\frac{x^s}{s}ds + O\left(\frac{x^b}{T}\right) + O\left(\frac{x^{\frac{1}{k}}logx}{T}\right) \\
&= \frac{1}{2\pi i}\int_{b-iT}^{b+iT} f(s)\frac{x^s}{s}ds + O\left(x^{\frac{1}{2k}+\varepsilon}\right).
\end{aligned}$$

Taking $a = \frac{1}{2k} + \frac{1}{logx}$, we have

$$\begin{aligned}
&\frac{1}{2\pi i}\left(\int_{b-iT}^{b+iT} + \int_{b+iT}^{a+iT} + \int_{a+iT}^{a-iT} + \int_{a-iT}^{b-iT}\right) = Res\left[f(s)\frac{x^s}{s}, \frac{1}{k}\right] \\
&= kx^{\frac{1}{k}} \prod_p \left[1 + \frac{k}{p^{\frac{1}{k}+k-2}(p-1)} + \frac{k-1}{p^{\frac{2}{k}+k-3}(p-1)} + \cdots + \frac{2}{p^{\frac{k-1}{k}}(p-1)}\right].
\end{aligned}$$

Note that the estimate

$$\left|\frac{1}{2\pi i}\int_{a-iT}^{a+iT} f(s)\frac{x^s}{s}\right| \ll x^{\frac{1}{2k}+\varepsilon};$$

$$\left|\frac{1}{2\pi i}\int_{a-iT}^{b-iT} f(s)\frac{x^s}{s}\right| \ll \frac{x^{\frac{1}{k}+\varepsilon}}{T} \ll x^{\frac{1}{2k}+\varepsilon};$$

and

$$\left|\frac{1}{2\pi i}\int_{a+iT}^{b+iT} f(s)\frac{x^s}{s}\right| \ll \frac{x^{\frac{1}{k}+\varepsilon}}{T} \ll x^{\frac{1}{2k}+\varepsilon},$$

we have

$$\sum_{n\leq x}\frac{d(a_k(n))}{\phi(a_k(n))}$$
$$= kx^{\frac{1}{k}}\prod_p\left[1+\frac{k}{p^{\frac{1}{k}+k-2}(p-1)}+\frac{k-1}{p^{\frac{2}{k}+k-3}(p-1)}+\cdots+\frac{2}{p^{\frac{k-1}{k}}(p-1)}\right]+O\left(x^{\frac{1}{2k}+\varepsilon}\right).$$

This completes the proof of the Theorem.

References

1. F. Smarandache, *Only problems, not Solutions*, Xiquan Publ. House, Chicago, 1993, pp. 26.
2. Tom M. Apostol, *Introduction to Analytic Number Theory*, Springer-Verlag, New York, 1976.
3. Pan Chengdong and Pan Chengbiao, *Foundation of Analytic Number Theory*, Science Press, Beijing, 1997, pp. 98.
4. "Smarandache Sequences" at http://www.gallup.unm.edu/~smarandache/snaqint.txt.
5. "Smarandache Sequences" at http://www.gallup.unm.edu/~smarandache/snaqint2.txt.
6. "Smarandache Sequences" at http://www.gallup.unm.edu/~smarandache/snaqint3.txt.