A NOTE ON THE PRIMES IN SMARANDACHE UNARY SEQUENCE

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Abstract

For any positive integer n, let p_n be the n-th prime, and let $u(n) = (10^{p_n} - 1)/9$. In this note we prove that if $p_n \equiv 1, 13$, or $19 \pmod{20}$, and $2p_n+1$ is also a prime, then $u(n)$ is not a prime.

For any positive integer n, let p_n be n-th prime, and let $u(n) = (10^{p_n} - 1)/9$. Then the sequence $U = \{u(n)\}_{n=1}^{\infty}$ is called the Smarandache unary sequence (see [2]).
It is an odd question that if U contain infinit many primes?
In this note we prove the following result:

Theorem. If $p_n \equiv 1, 13$, or $19 \pmod{20}$, and $2p_n+1$ is also a prime, then $u(n)$ is not a prime.

By using the above result, we see that both $u(12)$ and $u(15)$ are not primes.

Proof of Theorem. Let $q = 2p_n+1$. By Fermat's theorem (see[1], Theorem 71]), if q is a prime, then we have

(1) $$10^{q-1} \equiv 1 \pmod{q}.$$

From (1), we get

(2) $$(10^{p_n} + 1)(10^{p_n} - 1) \equiv 0 \pmod{q}.$$

Since q is a prime, we have either

(3) $$q \mid 10^{p_n} + 1$$

or

(4) $$q \mid 10^{p_n} - 1,$$

by (2).

We nou assume that p_n satisfies $p_n \equiv 1, 13$, or $19 \pmod{20}$.

Then p_n is an odd prime. Hence, if (3) holds, then we have

(5)

$$\left(\frac{-10}{q}\right) = 1,$$

where (-10/9) is the Legendre symbol. Since $q = 2p_n+1$,
we hawe $q \equiv 3 \pmod 4$ and $(-1/q) = -1$. Therefore, we obtain from (5) that

(6) $$(10/q) = (2/q)(5/q) = -1.$$

Hoewer, since $q \equiv 3, 27$, or $39 \pmod{40}$ if $p_n \equiv 1, 13$, or $19 \pmod{20}$ respectively, we have

(7) $$(2/q) = \begin{cases} -1, \\ 1, \end{cases} \quad (5/q) = \begin{cases} -1, & \text{if } q \equiv 3 \text{ or } 27 \pmod{40}; \\ 1, & \text{if } q \equiv 39 \pmod{40}. \end{cases}$$

We find for (7) that $(10/q) = 1$, which contradicts (6). It implies that (3) does not hold. Thus, by (4), we get

(8) $$q \mid 9u(n).$$

Notice that $q \pm 9$ and $1<q<u(n)$. We see from (8) that $q \mid u(n)$ and $u(n)$ is not a prime. The theorem is proved.

References:

1. G.H.Hardy and E.M.Wright, "An Introduction to the Theory of Numbers", Oxford University Press, 1937.
2. F.Iacobescu, "Smarandache partition type and other sequences", Bulletin of Pure and Applied Sciences, 16E (1997), No. 2, 237-240.