A Note on the Smarandache Near-To-Primorial Function

Charles Ashbacher
Decisionmark
200 2nd Ave. SE
Cedar Rapids, IA 52401 USA

In a brief paper passed on to the author[1], Michael R. Mudge used the definition of the Primorial function:

Definition: For p any prime, the Primorial function of p, p* is the product of all prime numbers less than or equal to p.

Examples:

3* = 2 * 3 = 6
11* = 2 * 3 * 5 * 7 * 11 = 2310

To define the Smarandache Near-To-Primorial Function SPr(n)

Definition: For n a positive integer, the Smarandache Near-To-Primorial Function SPr(n) is the smallest prime p such that either p* or p* + 1 or p* - 1 is divisible by n.

A table of initial values is also given

n	1	2	3	4	5	6	7	8	9	10	11 . . . 59
SPr(n)	2	2	2	5	3	3	3	5	?	5	11 . . . 13

and the following questions posed:

1) Is SPr(n) defined for all positive integers n?
2) What is the distribution of values of SPr(n)?
3) Is this problem fundamentally altered by replacing p* $\pm$ 1 by p* $\pm$ k for k = 3,5,...?

The purpose of this paper is to address these questions.

We start with a simple but important result that is presented in the form of a lemma.

Lemma 1: If the prime factorization of n contains more than one instance of a prime as a factor, then n cannot divide q* for q any prime.

Proof: Suppose that n contains at least one prime factor to a power greater than one, for reference purposes, call that prime p1. The list of prime factors of n contains a largest prime and we can call that prime p2. If we choose another arbitrary prime q, there are two cases to consider.

Case 1: $q < p2$. Then p2 cannot divide q*, as q* contains no instances of p2 by definition.

Case 2: $q \geq p2$. In this case, each prime factor of n will divide q*, but since p1 appears only once in q*, $p1^2$ cannot divide q*. Therefore, n cannot divide q* as well. □

We are now in a position to answer the first question.

Theorem 1: If n contains more than one instance of 2 as a factor, then SPr(n) does not exist.

Proof: Choose n to be a number having more than one instance of 2 as a factor. By lemma 1, there is no prime q such that n divides q*. Furthermore, since 2 is a prime, q* is always even. Therefore, $q^* \pm 1$ is always odd and n cannot evenly divide it. □

The negative answer to the first question also points out two errors in the Mudge table. SPr(4) and SPr(8) do not exist, and an inspection of the given values verifies this. The Primorial of 5 is 2*3*5 = 30 and no element in the set { 29,30,31 } is evenly divisible by 4.

By definition, the range of SPr(n) is a set of prime numbers. The obvious question is then whether the range of SPr(n) is in fact the set of all prime numbers, and we state the answer as a theorem.

Theorem 2: The range of SPr(n) is the set of all prime numbers.

Proof: The first few values are by inspection.

SPr(1) = 2, SPr(5) = 3, SPr(10) = 5

Choose an arbitrary prime $p > 5$ and construct the number p* - 1. Obviously, p* - 1 divides p* -1. It is also clear that there is no prime $q < p$ such that q*, $q^* + 1$ or $q^* - 1$ is divisible by p* -1. Therefore, SPr(p* - 1) $=$ p and p is in the range of SPr(n). □

Which answers the second question posed by M. Mudge.

It is easy to establish an algorithmic process to determine if SPr(n) is defined for values of n containing more than one instance of a prime greater than 2.

The first step is to prove another lemma.

Lemma 2: If n contains a prime p that appears more than once as a factor of n, and q is any prime $q \geq p$, then n does not divide $q^* \pm 1$.

Proof: Let n, p and q have the stated properties. Clearly, p divides q^* and since q is greater than 1, p cannot divide $q^* \pm 1$, forcing the conclusion that n cannot divide $q^* \pm 1$ as well. Combining this with lemma 1 gives the desired result. □

Corollary: If n contains some prime p more than once as a factor and SPr(n) exists, then the prime q such that n divides $q^* \pm 1$ must be less than p.

Proof: Clear. □

The next lemma deals with some of the instances where SPr(n) is defined.

Lemma 3: If $n = p_1 p_2 \ldots p_k$, where $k \geq 1$ and all p_i are primes, then SPr(n) is defined.

Proof: Let q denote the largest prime factor of n. By definition, q^* contains one instance of all primes less than or equal to q, so n must divide q^*. Given the existence of one such number, there must also be a minimal one. □

Combining all previous results, we can create a simple algorithm that can be used to determine if SPr(n) exists for any positive integer n.

Input: A positive integer n.
Output: Yes, if SPr(n) exists, No otherwise.

Step 1: Factor n into prime factors, $p_1 p_2 \ldots p_k$.
Step 2: If all primes appear to the first power, terminate with the message "Yes".
Step 3: If 2 appears to a power greater than 1, terminate with the message "No".
Step4: Set q = 2, the smallest prime.
Step 5: Compute $q^* + 1$ and $q^* - 1$.
Step 6: If n divides $q^* + 1$ or $q^* - 1$, terminate with the message "Yes".
Step 7: Increment q to the next largest prime.
Step 8: If $q \geq p$, terminate with the message "No".
Step 9: Goto step 5.

And this algorithm can be used to resolve the question mark in the Mudge table. Since 9 does not divide $2^* \pm 1$, SPr(9) is not defined. Furthermore, 3 to any power greater than 2 also cannot divide $2^* \pm 1$, so the conclusion is stronger in that SPr(n) is not defined for n any power of 3 greater than 3.

Note that modifications of this algorithm could be made so that it also returns the value of SPr(n) when defined.

These conclusions can be used to partially answer the third question. The conclusion of lemma 3 concerning all prime factors to the first power is unaffected. However, if $q \geq 3$ and q prime, then $q^* \pm 3$ is also divisible by 3, making solutions possible for higher powers of 3. Such results do indeed occur, as

$$3^* + 3 = 9$$

so that the modified SPr(9) = 9.

Reference

1. **The Smarandache Near-To-Primorial Function**, personal correspondence by Michael R. Mudge.