A NOTE ON THE SMARANDACHE PRIME PRODUCT SEQUENCE

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ABSTRACT

This paper gives some properties of the Smarandache prime product sequence, (P_n), definded by

$$P_n = 1+p_1 p_2 \ldots p_n, n \geq 1,$$

where (p_n) is the sequence of primes in their natural order.

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1. INTRODUCTION

Let $(p_n) = (p_1, p_2, \ldots)$ be the (infinite) sequence of primes in their natural numbers. The first few terms of the sequence are as follows:

$$p_1 = 2, p_2 = 3, p_3 = 5, p_4 = 7, p_5 = 11, p_6 = 13, p_7 = 17, p_8 = 19, p_9 = 23, p_{10} = 29.$$

Clearly, the sequence (p_n) is strictly increasing (in $n \geq 1$) with $p_n > p_1 p_2$ for all $n \geq 4$. Furthermore, $p_n > n$ for all $n \geq 1$.

The Smarandache prime product sequence, (p_n), is defined by (Smarandache [5])

$$P_n = 1+p_1 p_2 \ldots p_n, n \geq 1. \tag{1.1}$$

We note that the sequence (P_n) is strictly increasing (in $n \geq 1$), satisfying the following recursion formulas:

$$P_{n+1} = P_n + p_1 p_2 \ldots p_n(p_{n+1}-1), n \geq 1, \tag{1.2}$$

$$P_{n+1} = P_n p_{n+1} - (p_{n+1}-1), n \geq 1. \tag{1.3}$$

We also note that P_n is an odd (positive) integer for all $n \geq 1$; furthermore,

$$P_1 = 3, P_2 = 7, P_3 = 31, P_4 = 211, P_5 = 2311$$

are all primes, while the next five elements of the sequence (P_n) are all composites, since

$P_6 = 30031 = 59 \times 509,$

$P_7 = 510511 = 19 \times 97 \times 277,$

$P_8 = 9699691 = 347 \times 27953,$

$P_9 = 223092871 = 317 \times 703760,$

$P_{10} = 6469693231 = 331 \times 571 \times 34231.$

Some of the properties of the sequence (P_n) have been studied by Prakash [3], who conjectures that this sequence contains an infinite number of primes.

This note gives some properties of the sequence (P_n), some of which strengthens the corresponding result of Prakash [3]. This is done in §2 below, and show that for each $n \geq 1$, P_n is relatively prime to P_{n+1}. We conclude this paper with some remarks in the final §3.

2. MAIN RESULTS

We start with the following result which has been established by Majumdar [2] by induction on n (≥ 6), using the recurrence relationship (1.3).

Lemma 2.1: $P_n < (p_{n+1})^{n-2}$ for all $n \geq 6$.

Exploiting Lemma 2.1, Majumdar [2] has proved the following theorem which strengthens the corresponding result of Prakash [3].

Theorem 2.1: For each $n \geq 6$, P_n has at most n-3 prime factors (counting multiplicities).

Another property satisfied by the sequence (P_n) is given in Theorem 2.2. To prove the theorem, we would need the following results.

Lemma 2.2: For each $n \geq 1$, P_n is of the form $4k+3$ for some integer $k \geq 0$.

Proof: Since P_n is odd for all $n \geq 1$, it must be of the form $4k+1$ or $4k+3$ (see, for example, Shanks [4], pp. 4). But, P_n cannot be of the form $4k+1$, otherwise, from (1.1), we would have $p_1 p_2 \ldots p_n = 4k$,

that is, $4 | p_1 p_2 \ldots p_n$, which is absurd. Hence, P_n must be of the form $4k+3$. □

Lemma 2.3: (1) The product of two integers of the form $4k+1$ is an integer of the form $4k+1$, and in general, for any integer m > 0, $(4k+1)^m$ is again of the form $4k+1$,

(2) The product of two integers of the form $4k+3$ is an integer of the form $4k+1$, and the product of two integers, one of the form $4k+1$ and the other of the form $4k+3$, is integer of the form $4k+3$,

(3) For any integer $m > 0$, $(4k+3)^m$ is of the form $4k+1$ or $4k+3$ respectively according as m is even or odd.

Proof: Part (1) has been proved by Bolker ([1], Lemma 5.2, pp. 6). The proof of the remaining parts is similar. □

We now prove the following theorem.

Theorem 2.2: For each $n \geq 1$, P_n is never a square or higher power of any natural number (> 1).

Proof: If possible, let $P_n = N^2$ for some integer $N > 1$.

Now, since P_n is odd, N must be odd, and hence, N must be of the form $4k+1$ or $4k+3$ for some integer $k \geq 0$. But, in either case, by Lemma 2.3, $N^2 = P_n$ is of the form $4k+1$, contradicting Lemma 2.2. Hence, P_n cannot be a square of a natural number (> 1).

To prove the remaining part, let $P_n = N^l$ for some integers $N > 1$, $l \geq 3$. (*)

Without loss of generality, we may assume that l is a prime (if l is a composite number, let $l = rs$ where r is prime, and so $p_n = (N^s)^r$; setting $M = N^s$, we may proceed with this M in place of N). By Theorem 2.1, $1 < n$, and hence, 1 must be one of the primes $p_2, p_3, \ldots, p_n$. By Fermat's Little Theorem (Bolker [1], Theorem 9.8, pp. 16),

$$p_1 p_2 \ldots p_n = N^1 - 1 \equiv N - 1 \equiv 0 \pmod{1}.$$

Thus, $N = 1m+1$ for some integer $m > 0$,

and we get $p_1 p_2 \ldots p_n = (1m)^1 + \binom{1}{1}(1m)^{1-1} + \ldots + \binom{1}{1-1}(1m)$.

But the above expression shows that $1^2 1 p_1 p_2 \ldots p_n$, which is impossible.

Hence, the representation of P_n in the form (*) is not possible, which we intend to prove. □

Some more properties related to the sequence (P_n) are given in the following two lemmas. Lemma 2.4: For each $n \geq 1$, $(P_n, P_{n+1}) = 1$.

Proof: Any prime factor p of P_{n+1} satisfies the inequality $p > p_{n+1}$.

Now, if $p | P_n$, then from (1.3), we see that $p | (p_{n+1} - 1)$, which is absurd. Hence, all the prime factors of P_{n+1} are different from each of the prime factors of P_n, which proves the lemma. □

Lemma 2.5: For each $n \geq 1$, P_n and P_{n+2} have at most one prime factor in common.

Proof: Since $P_{n+2} - P_n = p_1 p_2 \ldots p_n \, (p_{n+1} p_{n+2} - 1)$,

any prime factor common to both P_n and P_{n+2} must divide $p_{n+1} p_{n+2} - 1$. Now, any prime factor of P_{n+2} is greater than p_{n+2}. Hence, it follows that P_n and p_{n+2} can have at most one prime factor in common, since otherwise, the product of the prime factors is greater than $(p_{n+2})^2$, which cannot divide $p_{n+1} p_{n+2} - 1 < (p_{n+2})^2$. □

From the proof of the above lemma we see that, if all the prime factors of $p_{n+1} p_{n+2} - 1$ are less than p_{n+2}, then $(P_n, P_{n+2}) = 1$. And generalizing the lemma, we have the following result: For any $n \geq 1$, and $i \geq 1$, P_n and P_{n+i} can have at most i-1 number of prime factors in common.

3. SOME REMARKS

We conclude this paper with the following remarks.

(1) The sequence (P_n) is well known, it is used in elementary texts on the Theory of Numbers (see, for example, Bolker [1]and Shanks [4] to prove the infinititude of the primes. Some of the properties of the sequence (P_n) have been studied by Prakash [3]. Theorem 2.1 improves one of the results of Prakash [3], while our proof of Theorem 2.2 is much simpler than that followed by Prakash [3]. The expressions for P_6, P_7, P_8, P_9 and P_{10} show that Theorem 2.1 is satisfied with tighter bounds, but we could not improve it further.

(2) By Lemma 2.3 we see that, of all the prime factors of P_n (which is at most n-3 in number for $n \geq 6$, by Theorem 2.1), an odd number of these must be of the form $4k$+3. In this connection, we note that, in case of P_6, one of the prime factors (namely, 59) is of the form $4k$+3, while the other is of the form $4k$+1; and in case of P_7, all the three prime factors are of the form $4k$+3.

(3) The Conjecture that the sequence (P_n) contains infinitely many primes, still remains an open problem.

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