A NOTE ON THE VALUES OF ZETA

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Carlitz considered the numbers $\eta_k(q)$ which are determined by

$$\eta_0(q)=0,\quad (q\eta(q)+1)^k-\eta_k(q)=\begin{cases}1 & \text{if } k=1\\ 0 & \text{if } k>1.\end{cases}$$

These numbers $\eta_k(q)$ induce Carlitz's kth q-Bernoulli numbers $\beta_k(q)=\beta_k$ as

$$\beta_0=1,\quad q(q\beta+1)^k-\beta_k=\begin{cases}1 & \text{if } k=1\\ 0 & \text{if } k>1\end{cases}$$

where we use the usual convention about replacing β^i by β_i $(i\geq 0)$.

Now, we modify the above number $\eta_m(q)$, that is,

$$B_0(q)=\frac{q-1}{\log q},\quad (qB(q)+1)^k-B_k(q)=\begin{cases}1 & \text{if } k=1\\ 0 & \text{if } k>1.\end{cases}$$

where we use the usual convention about replacing $B^i(q)$ by $B_i(q)$ $(i\geq 0)$.

In [1], I have constructed a complex q-series which is a q-analogue of Hurwitz's ζ-function. In this a short note, I will compute the values of zeta by using the q-series.

Let $F_q(t)$ be the generating function of $B_i(q)$:

$$F_q(t)=\sum_{k=0}^{\infty}B_k(q)\frac{t^k}{k!}\quad \text{for } q\in\mathbb{C} \text{ with } |q|<1.$$

This is the unique solution of the following q-difference equation:

$$F_q(t)=e^tF_q(qt)-t.$$

It is easy to see that

$$F_q(t)=-t\sum_{n=0}^{\infty}q^ne^{[n]t}+\frac{q-1}{\log q}e^{\frac{1}{1-q}t}.$$

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Thus we have:

$$B_k(q) = \frac{d^k}{dt^k}F_q(t)\bigg|_{t=0} = -k\sum_{n=0}^{\infty} q^n[n]^{k-1} + \frac{(-1)^k}{(q-1)^{k-1}}\frac{1}{\log q}.$$

Hence, we can define a q-analogue of the ζ-function as follows:

For $s \in \mathbb{C}$, define (see[1])

$$\zeta_q(s) = \sum_{n=1}^{\infty} \frac{q^n}{[n]^s} - \frac{1}{s-1}\frac{(1-q)^s}{\log q}.$$

Note that, $\zeta_q(s)$ is analytic continuation in $\mathbb{C}$ with only one simple pole at $s=1$ and

$$\zeta_q(1-k) = -\frac{B_k(q)}{k} \quad \text{where } k \text{ is any positive integer.}$$

Now, we define q-Bernoulli polynomial $B_n(x;q)$ as

$$B_n(x;q) = (q^x B(q) + [x])^n = \sum_{k=0}^{n} \binom{n}{k} q^{xi} B_k(q)[x]^{n-k}.$$

Let $T_q(x,t)$ be generating function of q-Bernoulli polynomials.

Note that

$$T_q(x,t) = F_q(q^x t)e^{[x]t}.$$

Thus

$$\begin{aligned} B_{k+1}(x;q) &= \frac{d^{k+1}}{dt^{k+1}}T_q(x,t)\bigg|_{t=0} \\ &= -(k+1)\sum_{n=0}^{\infty}([n]q^x + [x])^k q^{n+x} + \frac{q-1}{\log q}\left(\frac{1}{q-1}\right)^{k+1}. \end{aligned}$$

So, we can also define a q-analogue of the Hurwitz ζ-function as follows:

For $s \in \mathbb{C}$,(see [1])

$$\begin{aligned} \zeta_q(s,x) &= \sum_{n=0}^{\infty} \frac{q^{n+x}}{([n]q^x + [x])^s} - \frac{(1-q)^s}{\log q}\frac{1}{s-1} \\ &= \sum_{n=0}^{\infty} \frac{q^{n+x}}{[n+x]^s} - \frac{(1-q)^s}{\log q}\frac{1}{s-1}, \quad 0 < x \le 1. \end{aligned}$$

Note that, $\zeta_q(s,x)$ has an analytic continuation in $\mathbb{C}$ with only one simple pole at $s=1$.

Remark. $\zeta_q(s,x)$ is called q-analogue of Hurwitz ζ-function.

For $u \in \mathbb{C}$ with $u \neq 0, 1$, let $H_k(u : q)$ be q-Euler numbers (See [4]). It is known in [4] that $H_k(u : 1) = H_k(u)$ is the ordinary Euler number which is defined by

$$\frac{1-u}{e^t - u} = \sum_{k=0}^{\infty} H_k(u) \frac{t^k}{k!}.$$

In the case $u = -1, H_k(-1) = E_k$ is the classical Euler number is defined by

$$\frac{2}{e^t + 1} = \sum_{n=0}^{\infty} E_n \frac{t^k}{n!}.$$

Note that $E_{2k} = 0$ $(k \geq 1)$. In [4], $\ell_q(s, u)$ is defined by $\ell_q(s, u) = \sum_{n=1}^{\infty} \frac{u^{-n}}{[n]^s}$ and $\ell_q(-k, u) = \frac{u}{u-1} H_k(u : q)$ for $k > 1$.

Theorem 1. *For $s \in \mathbb{C}$, $f \in \mathbb{N} \backslash \{1\}$, we have*

(1) $\sum_{n=1}^{\infty} \frac{(-1)^n}{[n]^s} q^n = -\zeta_q^*(s) + \frac{2}{[2]^s} \zeta_{q^2}^*(s)$.

(2) $\zeta_q^*(s) = \frac{1}{[f]^s} \sum_{a=1}^{f} \zeta_{q^f}^*(s, \frac{a}{f})$, *where* $\zeta_q^*(s) = \sum_{n=1}^{\infty} \frac{q^n}{[n]^s}$.

It is easy to see that

$$\begin{aligned}
&\sum_{n=1}^{\infty} \frac{(-1)^n q^n}{[n]^{2k+1}} \sum_{j=0}^{\infty} \frac{\theta^{2j+1} [n]^{2j+1}}{(2j+1)!} + \frac{1}{\log q} \sum_{j=0}^{k-1} \frac{(1-q)^{2k-2j}}{2k-2j-1} \frac{\theta^{2j+1}}{(2j+1)!} \\
&\quad - \frac{1}{\log q} \sum_{j=0}^{k-1} \frac{\theta^{2j+1}}{(2j+1)!} \frac{(1-q^2)^{2k-2j}}{2k-2j-1} \frac{1}{[2]^{2k-2j}} \\
&= \sum_{j=0}^{k-1} \frac{(-1)^j \theta^{2j+1}}{(2j+1)!} \left(-\zeta_q(2k-2j) + \zeta_{q^2}(2k-2j) \frac{2}{[2]^{2k-2j}} \right) \\
&\quad - \frac{q}{1+q} \frac{\theta^{2k+1}}{(2k+1)!} (-1)^k + \sum_{j=k+1}^{\infty} \frac{\theta^{2j+1} (-1)^j}{(2j+1)!} \frac{q^{-1}}{1+q^{-1}} H_{2j-2k}(-q^{-1}, q).
\end{aligned}$$

If $q \to 1$, then we have

$$\begin{aligned}
\sum_{n=1}^{\infty} \frac{(-1)^n}{n^{2k+1}} \sin(n\theta) &= \sum_{j=0}^{k-1} \frac{(-1)^j}{(2j+1)!} \theta^{2j+1} \left(\frac{2}{2^{2k-2j}} - 1 \right) \\
&\qquad (-1)^{k-j+1} \frac{(2\pi)^{2k-2j}}{2 \cdot (2k-2j)!} B_{2k-2j} - \frac{1}{2} \frac{\theta^{2k+1}}{(2k+1)!} (-1)^k.
\end{aligned}$$

Let $k = 2$ and $\theta = \frac{\pi}{2}$. Then we have

$$\sum_{n=1}^{\infty} \frac{(-1)^n}{(2n+1)^5} = \pi^5 \left(\frac{1}{2^6 \cdot 5!} - \frac{B_2}{2^4 \cdot 3!} + \frac{7}{12} B_4 \right).$$

It is easy to see that

$$\sum_{n=1}^{\infty}\frac{(-1)^n}{(2n+1)^5}=2\sum_{n=1}^{\infty}\frac{1}{(4n-3)^5}+\sum_{n=1}^{\infty}\frac{1}{(2n)^5}-\sum_{n=1}^{\infty}\frac{1}{n^5}-1$$
$$=\frac{1}{2^9}\zeta(5,\frac{1}{4})-\frac{2^5-1}{2^5}\zeta(5)-1.$$

Thus we have

$$\zeta(5)-\frac{1}{2^4}\frac{1}{2^5-1}\zeta\left(5,\frac{1}{4}\right)-1=-2^5\pi^5\left(\frac{1}{2^6\cdot 5!}-\frac{B_2}{2^4\cdot 3!}+\frac{7}{12}B_4\right).$$

Therefore we obtain the following:

Proposition 2. $\zeta(5)-\frac{1}{2^4(2^5-1)}\zeta(5,\frac{1}{4})-1$ *is irrational.*

References

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