A proof of the non-existence of "Samma".

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Introduction: If $\prod_{i=1}^{k} p_i^{r_i}$ is the prime factorization of the natural number $n \geq 2$, then it is easy to verify that

$$S(n) = S(\prod_{i=1}^{k} p_i^{r_i}) = \max\{\ S(p_i^{r_i})\ \}_{i=1}^{k}.$$

From this formula we see that it is essensial to determine $S(p^r)$, where p is a prime and r is a natural number.

Legendres formula states that

$$n! = \prod_{i=1}^{k} p_i^{\sum_{m=1}^{\infty} [n/p_i^m]}. \tag{1}$$

The definition of the Smarandache function tells us that $S(p^r)$ is the least natural number such that $p^r \mid (S(p^r))!$. Combining this definition with (1), it is obvious that $S(p^r)$ must satisfy the following two inequalities:

$$\sum_{k=1}^{\infty} \left[\frac{S(p^r)-1}{p^k}\right] < \ r \ \leq \sum_{k=1}^{\infty} \left[\frac{S(p^r)}{p^k}\right]. \tag{2}$$

This formula (2) gives us a lower and an upper bound for $S(p^r)$, namely

$$(p-1)r + 1 \ \leq \ S(p^r) \ \leq \ pr. \tag{3}$$

It also implies that p divides $S(p^r)$, which means that

$$S(p^r) = p(r-i) \ \text{ for a particular } \ 0 \ \leq \ i \ \leq \ [\tfrac{r-1}{p}].$$

"Samma": Let $T(n) = 1 - \log(S(n)) + \sum_{i=2}^{n} \frac{1}{S(i)}$ for $n \geq 2$. I intend to prove that $\lim_{n\to\infty} T(n) = \infty$, i.e. "Samma" does not exists.

First of all we define the sequence $p_1 = 2$, $p_2 = 3$, $p_3 = 5$ and $p_n =$ the nth prime.

Next we consider the natural number p_m^n. Now (3) gives us that

$$\begin{aligned}
S(p_i^k) &\le p_i\,k \quad \forall\, i \in \{1,\ldots,m\} \text{ and } \forall\, k \in \{1,\ldots,n\} \\
&\Downarrow \\
\frac{1}{S(p_i^k)} &\ge \frac{1}{p_i\,k} \\
&\Downarrow \\
\sum_{i=1}^{m}\sum_{k=1}^{n} \frac{1}{S(p_i^k)} &\ge \sum_{i=1}^{m}\sum_{k=1}^{n} \frac{1}{p_i\,k} = \Big(\sum_{i=1}^{m} \frac{1}{p_i}\Big)\cdot\Big(\sum_{k=1}^{n} \frac{1}{k}\Big) \\
&\Downarrow
\end{aligned}$$

(4) $$\sum_{k=2}^{p_m^n} \frac{1}{S(k)} \ge \Big(\sum_{k=1}^{m} \frac{1}{p_k}\Big)\cdot\Big(\sum_{k=1}^{n} \frac{1}{k}\Big)$$

since $S(k) > 0$ for all $k \ge 2$, $p_a^b \le p_m^n$ whenever $a \le m$ and $b \le n$ and $p_a^b = p_c^d$ if and only if $a = c$ and $b = d$.

Futhermore $S(p_m^n) \le p_m\,n$, which implies that $-\log S(p_m^n) \ge -\log(p_m\,n)$ because $\log x$ is a strictly increasing function in the intervall $[2,\infty)$. By adding this last inequality and (4), we get

$$\begin{aligned}
T(p_m^n) &= 1 - \log(S(p_m^n)) + \sum_{i=2}^{p_m^n} \frac{1}{S(i)} \ge 1 - \log(p_m\,n) + \Big(\sum_{k=1}^{m} \frac{1}{p_k}\Big)\cdot\Big(\sum_{k=1}^{n} \frac{1}{k}\Big) \\
&\Downarrow \\
T(p_m^{p_m}) &\ge 1 - \log(p_m^2) + \Big(\sum_{k=1}^{m} \frac{1}{p_k}\Big)\cdot\Big(\sum_{k=1}^{p_m} \frac{1}{k}\Big) \quad (n = p_m) \\
&\Downarrow \\
T(p_m^{p_m}) &\ge 1 + 2\,\Big(-\log p_m + \sum_{k=1}^{p_m} \frac{1}{k}\Big) + \Big(-2 + \sum_{k=1}^{m} \frac{1}{p_k}\Big)\cdot\Big(\sum_{k=1}^{p_m} \frac{1}{k}\Big) \\
&\Downarrow \\
\lim_{m\to\infty} T(p_m^{p_m}) &\ge 1 + 2\cdot\lim_{m\to\infty}\Big(-\log p_m + \sum_{k=1}^{p_m} \frac{1}{k}\Big) + \lim_{m\to\infty}\Big[\Big(-2 + \sum_{k=1}^{m} \frac{1}{p_k}\Big)\cdot\Big(\sum_{k=1}^{p_m} \frac{1}{k}\Big)\Big] \\
&= 1 + 2\cdot\lim_{p_m\to\infty}\Big(-\log p_m + \sum_{k=1}^{p_m} \frac{1}{k}\Big) + \lim_{m\to\infty}\Big[\Big(-2 + \sum_{k=1}^{m} \frac{1}{p_k}\Big)\cdot\Big(\sum_{k=1}^{p_m} \frac{1}{k}\Big)\Big] \\
&= 1 + 2\gamma + \lim_{m\to\infty}\Big(-2 + \sum_{k=1}^{m} \frac{1}{p_k}\Big)\cdot\lim_{p_m\to\infty}\Big(\sum_{k=1}^{p_m} \frac{1}{k}\Big) \quad (\gamma = \text{Euler's constant}) \\
&= \infty
\end{aligned}$$

since both $\sum_{k=1}^{t} \frac{1}{k}$ and $\sum_{k=1}^{t} \frac{1}{p_k}$ diverges as $t \to \infty$. In other words, $\lim_{n\to\infty} T(n) = \infty$. □