A recurrence formula for prime numbers using the Smarandache or Totient functions

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Abstract

In this paper we report a recurrence formula to obtain the n-th prime in terms of (n-1)th prime and as a function of Smarandache or Totient function.

In [1] the Smarandache Prime Function is defined as follows:

$$P:\ N \rightarrow (0,1)$$

$$\text{where: } P(n) = \begin{cases} 0 & \text{if n is prime} \\ 1 & \text{if n is composite} \end{cases}$$

This function can be used to determine the number of primes $\pi(N)$ less or equal to some integer N and to determine a recurrence formula to obtain the n-th prime starting from the (n-1)th one.

In fact:

$$\pi(N) = \sum_{i=1}^{N} (1 - P(i))$$

and

$$p_{n+1} = 1 + p_n + \sum_{j=p_n+1}^{2p_n} \prod_{i=p_n+1}^{j} P(i)$$

where p_n is the n-th prime.

The first equation is obvious because $(1-P(i))$ is equal to 1 every time i is a prime. Let's prove the second one.
Since $p_{n+1} < 2p_n$ [2] where p_{n+1} and p_n are the (n+1)th and n-th prime respectively, the following equality holds [3]:

$$\sum_{j=p_n+1}^{2p_n} \prod_{i=p_n+1}^{j} P(i) = \sum_{j=p_n+1}^{p_{n+1}-1} \prod_{i=p_n+1}^{j} P(i) + \sum_{j=p_{n+1}}^{2p_n} \prod_{i=p_n+1}^{j} P(i) = \sum_{j=p_n+1}^{p_{n+1}-1} \prod_{i=p_n+1}^{j} P(i)$$

because $\sum_{j=p_{n+1}}^{2p_n} \prod_{i=p_n+1}^{j} P(i) = 0$ being $P(p_{n+1}) = 0$ by definition.

As:

$$\sum_{j=p_n+1}^{p_{n+1}-1} \prod_{i=p_n+1}^{j} P(i) = \sum_{j=p_n+1}^{p_{n+1}-1} 1 = (p_{n+1}-1)-(p_n+1)+1 = p_{n+1}-p_n-1$$

we get:

$$p_{n+1} = 1 + p_n + \sum_{j=p_n+1}^{2p_n} \prod_{i=p_n+1}^{j} P(i) \quad \text{q.e.d}$$

According to this result we can obtain p_{n+1} once we know p_n and $P(i)$.

We report now two expressions for $P(i)$ using the Smarandache function S(n) [4] and the well known Totient function $\varphi(n)$ [5].

- $P(i) = 1 - \left\lfloor \frac{S(i)}{i} \right\rfloor$ for $i > 4$

- $P(i) = 1 - \left\lfloor \frac{\varphi(i)}{i-1} \right\rfloor$ for $i > 1$

where $\lfloor n \rfloor$ is the floor function [6]. Let's prove now the first equality.

By definition of Smarandache function $S(i) = i \quad for \ i \in P \cup \{1, 4\}$ where P is the set of prime numbers [6]. Then $\left\lfloor \frac{S(i)}{i} \right\rfloor$ is equal to 1 if i is a prime number and equal to zero for all composite > 4 being $S(i) \le i$ [4].

About the second equality we can notice that by definition $\varphi(n) < n \quad for \ n > 1$ and $\varphi(n) = n-1$ if and only if n is a prime number [5]. So $\varphi(n) \le n-1 \quad for \ n > 1$ and this implies that $\left\lfloor \frac{\varphi(i)}{i-1} \right\rfloor$ is equal to 1 if i is a prime number and equal to zero otherwise..

Then:

$$p_{n+1} = 1 + p_n + \sum_{j=p_n+1}^{2p_n} \prod_{i=p_n+1}^{j} \left(1 - \left\lfloor \frac{S(i)}{i} \right\rfloor\right) \quad for \ n > 2$$

and

$$p_{n+1} = 1 + p_n + \sum_{j=p_n+1}^{2p_n} \prod_{i=p_n+1}^{j} \left(1 - \left\lfloor \frac{\varphi(i)}{i-1} \right\rfloor\right) \quad for \ n \ge 1$$

According to the result obtained in [7] for the Smarandache function:

$$S(i) = i + 1 - \left\lfloor \sum_{k=1}^{i} i^{-(i \cdot \sin(k! \frac{\pi}{i}))^2} \right\rfloor$$

and in [3] for the following function:

$$\left\lfloor \frac{i}{k} \right\rfloor - \left\lfloor \frac{i-1}{k} \right\rfloor = \begin{cases} 1 & \text{if k divide i} \\ 0 & \text{if k doesn't divide i} \end{cases}$$

the previous recurrence formulas can be further semplified as follows:

$$p_{n+1} = 1 + p_n + \sum_{j=p_n+1}^{2p_n} \prod_{i=p_n+1}^{j} \left(1 - \left\lfloor \frac{\left(i + 1 - \left\lfloor \sum_{k=1}^{i} i^{-(i \cdot \sin(k! \cdot \frac{\pi}{i}))^2} \right\rfloor \right)}{i} \right\rfloor \right) \quad \text{for } n > 2$$

and

$$p_{n+1} = 1 + p_n + \sum_{j=p_n+1}^{2p_n} \prod_{i=p_n+1}^{j} \left(1 - \left\lfloor \frac{\sum_{k=1}^{i} \left(1 - \left(\left\lfloor \frac{i}{k} \right\rfloor - \left\lfloor \frac{i-1}{k} \right\rfloor \right) \right)}{i-1} \right\rfloor \right) \quad \text{for } n \geq 1$$

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