

# A result obtained using Smarandache Function

Sebastián Martín Ruiz

21 November 1998

Smarandache Function is defined as followed:

$S(m)$  = The smallest positive integer so that  $S(m)!$  is divisible by  $m$ . [1]

Let's see the value which such function takes for  $m = p^n$  with  $n$  integer,  $n \geq 2$  and  $p$  prime number. To do so a Lemma is required.

Lemma 1  $\forall m, n \in \mathbb{N} \quad m, n \geq 2$

$$m^n = E \left[ \frac{m^{n+1} - m^n + m}{m} \right] + E \left[ \frac{m^{n+1} - m^n + m}{m^2} \right] + \dots + E \left[ \frac{m^{n+1} - m^n + m}{m^{E[\log_m(m^{n+1} - m^n + m)]}} \right]$$

Where  $E(x)$  gives the greatest integer less than or equal to  $x$ .

Demonstration:

Let's see in the first place the value taken by  $E[\log_m(m^{n+1} - m^n + m)]$ .

If  $n \geq 2$ :  $m^{n+1} - m^n + m < m^{n+1}$  and therefore  $\log_m(m^{n+1} - m^n + m) < \log_m m^{n+1} = n + 1$ . As a result  $E[\log_m(m^{n+1} - m^n + m)] < n + 1$ .

And if  $m \geq 2$ :  $mm^n \geq 2m^n \Rightarrow m^{n+1} \geq 2m^n \Rightarrow m^{n+1} + m \geq 2m^n \Rightarrow m^{n+1} - m^n + m \geq m^n \Rightarrow \log_m(m^{n+1} - m^n + m) \geq \log_m m^n = n \Rightarrow E[\log_m(m^{n+1} - m^n + m)] \geq n$

As a result:  $n \leq E[\log_m(m^{n+1} - m^n + m)] < n + 1$  therefore:

$$E[\log_m(m^{n+1} - m^n + m)] = n \quad \text{if } n, m \geq 2$$

Now let's see the value which it takes for  $1 \leq k \leq n$ :  $E \left[ \frac{m^{n+1} - m^n + m}{m^k} \right]$

$$E \left[ \frac{m^{n+1} - m^n + m}{m^k} \right] = E \left[ m^{n+1-k} - m^{n-k} + \frac{1}{m^{k-1}} \right]$$

If  $k = 1$ :  $E \left[ \frac{m^{n+1} - m^n + m}{m} \right] = m^n - m^{n-1} + 1$

If  $1 < k \leq n$ :  $E \left[ \frac{m^{n+1} - m^n + m}{m^k} \right] = m^{n+1-k} - m^{n-k}$

Let's see what is the value of the sum:

$$\begin{array}{cccccccc}
 k = 1 & m^n & -m^{n-1} & \dots & \dots & \dots & \dots & +1 \\
 k = 2 & & m^{n-1} & -m^{n-2} & & & & \\
 k = 3 & & & m^{n-2} & -m^{n-3} & & & \\
 \vdots & & & & & & & \\
 k = n-1 & & & & & & m^2 & -m \\
 k = n & & & & & & & m-1
 \end{array}$$

Therefore:

$$\sum_{k=1}^n E \left[ \frac{m^{n+1} - m^n + m}{m^k} \right] = m^n \quad m, n \geq 2$$

Proposition 1  $\forall p$  prime number  $\forall n \geq 2$  :

$$S(p^{p^n}) = p^{n+1} - p^n + p$$

Demonstration:

Having  $e_p(k)$  = exponent of the prime number  $p$  in the prime numbers decomposition of  $k$ .

We get:

$$e_p(k!) = E\left(\frac{k}{p}\right) + E\left(\frac{k}{p^2}\right) + E\left(\frac{k}{p^3}\right) + \dots + E\left(\frac{k}{p^{E(\log_p k)}}\right)$$

And using the Lemma we have:

$$e_p[(p^{n+1} - p^n + p)!] = E\left[\frac{p^{n+1} - p^n + p}{p}\right] + E\left[\frac{p^{n+1} - p^n + p}{p^2}\right] + \dots + E\left[\frac{p^{n+1} - p^n + p}{m^{E[\log_p(p^{n+1} - p^n + p)]}}\right] = p^n$$

Therefore:

$$\frac{(p^{n+1} - p^n + p)!}{p^{p^n}} \in \mathbf{N} \quad \text{and} \quad \frac{(p^{n+1} - p^n + p - 1)!}{p^{p^n}} \notin \mathbf{N}$$

And:

$$S(p^{p^n}) = p^{n+1} - p^n + p$$

### References:

[1] C. Dumitrescu and R. Müller: *To Enjoy is a Permanent Component of Mathematics*. SMARANDACHE NOTIONS JOURNAL Vol. 9, No. 1-2,(1998) pp 21-26.

Author:

Sebastián Martín Ruiz. Avda. de Regla 43. CHIPIONA (CADIZ) 11550 SPAIN.