ABOUT THE SMARANDACHE COMPLEMENTARY CUBIC FUNCTION

by Marcela Popescu and Mariana Nicolescu

DEFINITION. *Let* $g:N^* \to N^*$ *be a numerical function defined by* $g(n) = k$, *where* k *is the smallest natural number such that* nk *is a perfect cube:* $nk = s^3, s \in N^*$.

Examples: 1) $g(7)=49$ because 49 is the smallest natural number such that $7 \cdot 49 = 7 \cdot 7^2 = 7^3$;

2) $g(12) = 18$ because 18 is the smallest natural number such that $12 \cdot 18 = (2^2 \cdot 3) \cdot (2 \cdot 3^2) = 2^3 \cdot 3^3 = (2 \cdot 3)^3$;

3) $g(27) = g(3^3) = 1$;

4) $g(54) = g(2 \cdot 3^3) = 2^2 = g(2)$.

PROPERTY 1. *For every* $n \in N^*$, $g(n^3) = 1$ *and for every prime* p *we have* $g(p) = p^2$.

PROPERTY 2. *Let* n *be a composite natural number and* $n = p_{i_1}^{\alpha_{i_1}} \cdot p_{i_2}^{\alpha_{i_2}} \cdot \ldots \cdot p_{i_r}^{\alpha_{i_r}}$, $0 < p_{i_1} < p_{i_2} < \cdots < p_{i_r}$, $\alpha_{i_1}, \alpha_{i_2}, \ldots, \alpha_{i_r} \in N^*$ *its prime factorization. Then* $g(n) = p_{i_1}^{d(\overline{\alpha_{i_1}})} \cdot p_{i_2}^{d(\overline{\alpha_{i_2}})} \cdot \ldots \cdot p_{i_r}^{d(\overline{\alpha_{i_r}})}$, *where* $\overline{\alpha_{i_j}}$ *is the remainder of the division of* α_{i_j} *by 3 and* $d:\{0,1,2\} \to \{0,1,2\}$ *is the numerical function defined by* $d(0) = 0, d(1) = 2$ *and* $d(2) = 1$.

If we take into account of the above definition of the function g, it is easy to prove the above properties.

OBSERVATION: $d(\overline{\alpha_{i_j}}) = \overline{3 - \overline{\alpha_{i_j}}}$, for every $\alpha_{i_j} \in N^*$, and in the sequel we use this writing for its simplicity.

REMARK 1. *Let* $m \in N^*$ *be a fixed natural number. If we consider now the numerical function* $\tilde{g}:N^* \to N^*$ *defined by* $\tilde{g}(n) = k$, *where* k *is the smallest natural number such that* $nk = s^m, s \in N^*$, *then we can observe that* $\tilde{g}$ *generalize the function* g, *and we also have:* $\tilde{g}(n^m) = 1$, $\forall n \in N^*$, $\tilde{g}(p) = p^{m-1}$, $\forall p$ *prime and* $\tilde{g}(n) = p_{i_1}^{\overline{m-\overline{\alpha_{i_1}}}} \cdot p_{i_2}^{\overline{m-\overline{\alpha_{i_2}}}} \cdot \ldots \cdot p_{i_r}^{\overline{m-\overline{\alpha_{i_r}}}}$, *where* $n = p_{i_1}^{\alpha_{i_1}} \cdot p_{i_2}^{\alpha_{i_2}} \cdot \ldots \cdot p_{i_r}^{\alpha_{i_r}}$ *is the prime factorization of n and* $\overline{\alpha_{i_j}}$ *is the remainder of the division of* α_{i_j} *by m, therefore the both above properties holds for* $\tilde{g}$, *too.*

REMARK 2. *Because* $1 \le g(n) \le n^2$, *for every* $n \in N^*$, *we have:* $\frac{1}{n} \le \frac{g(n)}{n} \le n$, *thus* $\sum_{n \ge 1} \frac{g(n)}{n}$ *is a divergent serie.*

In a similar way, using that we have $1 \le \tilde{g}(n) \le n^{m-1}$ *for every* $n \in N^*$, *it results that* $\sum_{n\ge1} \frac{\tilde{g}(n)}{n}$ *is also divergent.*

PROPERTY 3. *The function* $g: N^* \to N^*$ *is multiplicative:* $g(x \cdot y) = g(x) \cdot g(y)$ *for every* $x, y \in N^*$ *with* $(x, y) = 1$.

Proof. For $x = 1 = y$ we have $(x, y) = 1$ and $g(1 \cdot 1) = g(1) \cdot g(1)$. Let $x = p_{i_1}^{\alpha_{i_1}} \cdot p_{i_2}^{\alpha_{i_2}} \cdot \ldots \cdot p_{i_r}^{\alpha_{i_r}}$ and $y = q_{j_1}^{\beta_{j_1}} \cdot q_{j_2}^{\beta_{j_2}} \cdot \ldots \cdot q_{j_s}^{\beta_{j_s}}$ be the prime factorization of x and y, repectively, so that $x \cdot y = 1$.

Because $(x, y) = 1$ we have $p_{i_h} \ne q_{j_k}$, for every $h = \overline{1, r}$ and $k = \overline{1, s}$.

Then $g(x \cdot y) = p_{i_1}^{\overline{\overline{3-\alpha_{i_1}}}} \cdot p_{i_2}^{\overline{\overline{3-\alpha_{i_2}}}} \cdot \ldots \cdot p_{i_r}^{\overline{\overline{3-\alpha_{i_r}}}} \cdot q_{j_1}^{\overline{\overline{3-\beta_{j_1}}}} \cdot q_{j_2}^{\overline{\overline{3-\beta_{j_2}}}} \cdot \ldots \cdot q_{j_s}^{\overline{\overline{3-\beta_{j_s}}}} = g(x) \cdot g(y)$.

REMARK 3. *The property holds also for the function* $\tilde{g}: \tilde{g}(x \cdot y) = \tilde{g}(x) \cdot \tilde{g}(y)$, *where* $(x, y) = 1$.

PROPERTY 4. *If* $(x, y) = 1$, x *and* y *are not perfect cubes and* x,y>1, then *the equation* $g(x) = g(y)$ *has not natural solutions.*

Proof. Let $x = \prod_{h=1}^{r} p_{i_h}^{\alpha_{i_h}}$ and $y = \prod_{k=1}^{s} q_{j_k}^{\beta_{j_k}}$ (where $p_{i_h} \ne q_{j_k}, \forall h = \overline{1, r}, k = \overline{1, s}$, because $(x, y) = 1$) be their prime factorizations. Then $g(x) = \prod_{h=1}^{r} p_{i_h}^{\overline{\overline{3-\alpha_{i_h}}}}$ and $g(y) = \prod_{k=1}^{s} q_{j_k}^{\overline{\overline{3-\beta_{j_k}}}}$ and there exist at least $\overline{\alpha}_{i_h} \ne 0$ and $\overline{\beta}_{j_k} \ne 0$ (because x and y are not perfect cubes), therefore $1 \ne p_{i_h}^{\overline{\overline{3-\alpha_{i_h}}}} \ne q_{j_k}^{\overline{\overline{3-\beta_{j_k}}}} \ne 1$, so $g(x) \ne g(y)$.

CONSEQUENCE 1. *The equation* $g(x) = g(x+1)$ *has not natural solutions because for* $x \ge 1$, *x and x+1 are not both perfect cubes and* $(x, x+1) = 1$.

REMARK 4. *The property and the consequence is also true for the function* $\tilde{g}$: if $(x, y) = 1$, $x > 1$, $y > 1$, *and it does not exist* $a, b \in N^*$ *so that* $x = a^m$, $y = b^m$ *(where m is fixed and has the above significance), then the equation* $\tilde{g}(x) = \tilde{g}(y)$ *has not natural solutions; the equation* $\tilde{g}(x) = \tilde{g}(x+1)$, $x \ge 1$ *has not natural solutions, too.*

It is easy to see that the proofs are similary, but in this case we denote by $\overline{\alpha}_{ij} = \alpha_{ij}$ *(mod m) and we replace* $\overline{\overline{3-\alpha_{i_1}}}$ *by* $\overline{\overline{m-\alpha_{i_j}}}$.

PROPERTY 5. *We have* $g(x \cdot y^3) = g(x)$, *for every* $x, y \in N^*$.

Proof. If $(x, y) = 1$, then $(x, y^3) = 1$ and using property 1 and property 3, we have: $g(x \cdot y^3) = g(x) \cdot g(y^3) = g(x)$.

If $(x,y)=1$ we can write: $x=\prod_{h=1}^{r} p_{i_h}^{\alpha_{i_h}} \cdot \prod_{t=1}^{n} d_{l_t}^{\alpha_{l_t}}$ and $y=\prod_{k=1}^{s} q_{j_k}^{\beta_{j_k}} \cdot \prod_{t=1}^{n} d_{l_t}^{\beta_{l_t}}$ where $p_{i_h} \neq d_{l_t}; q_{j_k} = d_{l_t}, p_{i_h} \neq q_{j_k}; \forall h=\overline{1,r}, k=\overline{1,s}$, $t=\overline{1,n}$. We have $g(x\cdot y^3)=g(\prod_{h=1}^{r} p_{i_h}^{\alpha_{i_h}} \cdot \prod_{t=1}^{n} d_{l_t}^{\alpha_{l_t}} \cdot \prod_{k=1}^{s} q_{j_k}^{3\beta_{j_k}} \cdot \prod_{t=1}^{n} d_{l_t}^{3\beta_{l_t}}) = g(\prod_{h=1}^{r} p_{i_h}^{\alpha_{i_h}} \cdot \prod_{k=1}^{s} q_{j_k}^{3\beta_{j_k}} \cdot \prod_{t=1}^{n} d_{l_t}^{\alpha_{l_t}+3\beta_{l_t}}) = g(\prod_{h=1}^{r} p_{i_h}^{\alpha_{i_h}} \cdot \prod_{k=1}^{s} q_{j_k}^{3\beta_{j_k}}) \cdot g(\prod_{t=1}^{n} d_{l_t}^{\alpha_{l_t}+3\beta_{l_t}}) =$

$= \prod_{h=1}^{r} p_{i_h}^{\overline{3-\overline{\alpha_{i_h}}}} \cdot \prod_{k=1}^{s} q_{j_k}^{\overline{3-\overline{3\beta_{j_k}}}} \cdot \prod_{t=1}^{n} d_{l_t}^{\overline{3-\overline{\alpha_{l_t}+3\beta_{l_t}}}} = \prod_{h=1}^{r} p_{i_h}^{\overline{3-\overline{\alpha_{i_h}}}} \cdot \prod_{t=1}^{n} d_{l_t}^{\overline{3-\overline{\alpha_{l_t}}}} = g(\prod_{h=1}^{r} p_{i_h}^{\alpha_{i_h}}) \cdot g(\prod_{t=1}^{n} d_{l_t}^{\alpha_{l_t}}) =$

$= g(\prod_{h=1}^{r} p_{i_h}^{\alpha_{i_h}} \cdot \prod_{t=1}^{n} d_{l_t}^{\alpha_{l_t}}) = g(x).$

We used that $(\prod_{h=1}^{r} p_{i_h}^{\alpha_{i_h}}, \prod_{t=1}^{n} d_{l_t}^{\alpha_{l_t}})=1$ and $(\prod_{h=1}^{r} p_{i_h}^{\alpha_{i_h}} \cdot \prod_{k=1}^{s} q_{j_k}^{3\beta_{j_k}}, \prod_{t=1}^{n} d_{l_t}^{\alpha_{l_t}-3\beta_{l_t}})=1$ and the above properties.

REMARK 5. *It is easy to see that we also have* $\tilde{g}(x\cdot y^m)=\tilde{g}(x)$, *for every* $x,y \in N^*$.

OBSERVATION . *If* $\frac{x}{y}=\frac{u^3}{v^3}$, *where* $\frac{u}{v}$ *is a simplified fraction, then g(x)=g(y). It is easy to prove this because* $x=kn^3$ *and* $y=kv^3$, *and using the above property we have:*

$$g(x)=g(k\cdot u^3)=g(k)=g(k\cdot v^3)=g(y)$$

OBSERVATION. If $\frac{x}{y}=\frac{u^m}{v^m}$ where $\frac{u}{v}$ is a simplified fraction, then, using remark 5, we have $\tilde{g}(x)=\tilde{g}(y)$, too.

CONSEQUENCE 2. *For every* $x \in N^*$ *and* $n \in N$,

$$g(x^n)=\begin{cases} 1, \text{ if } n=3k; \\ g(x), \text{ if } n=3k+1; \\ g^2(x), \text{ if } n=3k+2,\ k\in N, \end{cases}$$

where $g^2(x)=g(g(x))$.

Proof. If n=3k, then x^n is a perfect cube, therefore $g(x^n)=1$.
If n=3k+1, then $g(x^n)=g(x^{3k}\cdot x)=g(x^{3k})\cdot g(x)=g(x)$.
If n=3k+2, then $g(x^n)=g(x^{3k}\cdot x^2)=g(x^{3k})\cdot g(x^2)=g(x^2)$.

PROPERTY 6. $g(x^2)=g^2(x)$, for every $x\in \mathbf{N}^*$.

Proof. Let $x=\prod_{h=1}^{r} p_{i_h}^{\alpha_{i_h}}$ be the prime factorization of x. Then $g(x^2)=g(\prod_{h=1}^{r} p_{i_h}^{2\alpha_{i_h}})=\prod_{h=1}^{r} p_{i_h}^{\overline{3-\overline{2\alpha_{i_h}}}}$ and $g^2(x)=g(g(x))=g(\prod_{h=1}^{r} p_{i_h}^{\overline{3-\overline{\alpha_{i_h}}}})=\prod_{h=1}^{r} p_{i_h}^{\overline{3-\overline{3-\overline{\alpha_{i_h}}}}}$, but it is easy to observe that $\overline{3-\overline{2\alpha_{i_h}}}=\overline{3-\overline{3-\overline{\alpha_{i_h}}}}$, because for :

$\overline{\alpha_{i_h}} = 0 \qquad \overline{3-\overline{2\alpha_{i_h}}} = \overline{3-0} = 0 \qquad$ and $\quad \overline{3-\overline{3-\overline{\alpha_{i_h}}}} = \overline{3-\overline{3-0}} = \overline{3-0} = 0$

$\overline{\alpha_{i_h}} = 1 \qquad \overline{3-\overline{2\alpha_{i_h}}} = \overline{3-2} = 1 \qquad$ and $\quad \overline{3-\overline{3-\overline{\alpha_{i_h}}}} = \overline{3-\overline{3-1}} = \overline{3-2} = 1$

$\overline{\alpha_{i_h}} = 2 \qquad \overline{3-\overline{2\alpha_{i_h}}} = \overline{3-\overline{4}} = \overline{3-1} = 2$ and $\overline{3-\overline{3-\overline{\alpha_{i_h}}}} = \overline{3-\overline{3-2}} = \overline{3-1} = 2,$

therefore $g(x^2) = g^2(x)$.

REMARK 6. *For the function* $\tilde{g}$ *is not true that* $\tilde{g}(x^2) = \tilde{g}^2(x)$, $\forall x \in \mathbf{N}^*$. *For example, for* $m=5$ *and* $x = 3^2$, $\tilde{g}(x^2) = \tilde{g}(3^4) = 3$ *while* $\tilde{g}(\tilde{g}(3^2)) = \tilde{g}(3^3) = 3^2$.

More generally $\tilde{g}(x^k) = \tilde{g}^k(x)$, $\forall x \in \mathbf{N}^*$ is not true. But for particular values of m, k and x the above equality is possible to be true. For example for $m = 6$, $x = 2^2$ and $k = 2$: $\tilde{g}(x^2) = \tilde{g}(2^4) = 2^2$ and $\tilde{g}^2(x) = \tilde{g}(\tilde{g}(2^2)) = \tilde{g}(2^4) = 2^2$.

REMARK 6'. *a)* $\tilde{g}(x^{m-1}) = \tilde{g}^{m-1}(x)$ *for every* $x \in \mathbf{N}^*$ *iff m is an odd number, because we have* $\overline{m-\overline{(m-1)\alpha_{i_h}}} = \underbrace{\overline{m-\overline{m-\ldots-\overline{m-\overline{\alpha_{i_h}}}}}}_{m-1 \text{ times}}$, *for every* $\alpha_{i_h} \in \mathbf{N}$.

Example: *For* $m = 5$, $\tilde{g}(x^4) = \tilde{g}^4(x)$, *for every* $x \in \mathbf{N}^*$.

b) $\tilde{g}(x^{m-1}) = \tilde{g}^{m}(x)$, *for every* $x \in \mathbf{N}^*$ *iff m is an even number, because we have* $\overline{m-\overline{(m-1)\alpha_{i_h}}} = \underbrace{\overline{m-\overline{m-\ldots-\overline{m-\overline{\alpha_{i_h}}}}}}_{m \text{ times}}$, *for every* $\alpha_{i_h} \in \mathbf{N}$.

Example: *For* $m = 4$, $\tilde{g}(x^3) = \tilde{g}^4(x)$, *for every* $x \in \mathbf{N}^*$.

PROPERTY 7. *For every* $x \in \mathbf{N}^*$ *we have* $g^3(x) = g(x)$.

Proof. Let $x = \prod_{h=1}^{r} p_{i_h}^{\alpha_{i_h}}$ be the prime factorization of x. We saw that $g(x) = \prod_{h=1}^{r} p_{i_h}^{\overline{3-\overline{\alpha_{i_h}}}}$ and

$$g^3(x) = g(g^2(x)) = g\left(\prod_{h=1}^{r} p_{i_h}^{\overline{3-\overline{3-\overline{\alpha_{i_h}}}}}\right) = \prod_{h=1}^{r} p_{i_h}^{\overline{3-\overline{3-\overline{3-\overline{\alpha_{i_h}}}}}}.$$

But $\overline{3-\overline{\alpha_{i_h}}} = \overline{3-\overline{3-\overline{3-\overline{\alpha_{i_h}}}}}$, for every $\alpha_{i_h} \in \mathbf{N}$, because for:

$\overline{\alpha_{i_h}} = 0 \qquad \overline{3-\overline{\alpha_{i_h}}} = 0$ and $\overline{3-\overline{3-\overline{3-\overline{\alpha_{i_h}}}}} = \overline{3-\overline{3-\overline{3-0}}} = \overline{3-\overline{3-0}} = \overline{3-0} = 0$

$\overline{\alpha_{i_h}} = 1 \qquad \overline{3-\overline{\alpha_{i_h}}} = 2$ and $\overline{3-\overline{3-\overline{3-\overline{\alpha_{i_h}}}}} = \overline{3-\overline{3-\overline{3-1}}} = \overline{3-\overline{3-2}} = \overline{3-1} = 2$

$\overline{\alpha_{i_h}} = 2 \qquad \overline{3-\overline{\alpha_{i_h}}} = 1$ and $\overline{3-\overline{3-\overline{3-\overline{\alpha_{i_h}}}}} = \overline{3-\overline{3-\overline{3-2}}} = \overline{3-\overline{3-1}} = \overline{3-2} = 1,$

therefore $g^3(x) = g(x)$, for every $x \in \mathbf{N}^*$.

REMARK 7. *For every* $x\in N^*$ *we have* $\tilde{g}^3(x)=\tilde{g}(x)$ because $\overline{m-\overline{\alpha_{i_h}}}=\overline{m-\overline{m-\overline{m-\overline{\alpha_{i_h}}}}}$, for every $\alpha_{i_h}\in\mathbf{N}$. For $\overline{\alpha_{i_h}}=a\in\{1,\ldots,m-1\}=A$, we have $\overline{m-\overline{\alpha_{i_h}}}=m-a\in A$, therefore $\overline{m-\overline{m-\overline{\alpha_{i_h}}}}=\overline{m-(m-a)}=\bar{a}=a$, so that $\overline{m-\overline{m-\overline{m-\overline{\alpha_{i_h}}}}}=\overline{m-a}=\overline{m-\overline{\alpha_{i_h}}}$, which is also true for $\overline{\alpha_{i_h}}=0$, therefore it is true for every $\alpha_{i_h}\in N^*$.

PROPERTY 8. *For every* $x,y\in N^*$ *we have* $g(x\cdot y)=g^2(g(x)\cdot g(y))$.

Proof. Let $x=\prod_{h=1}^{r}p_{i_h}^{\alpha_{i_h}}\cdot\prod_{t=1}^{n}d_{l_t}^{\alpha_{l_t}}$ and $y=\prod_{k=1}^{s}q_{j_k}^{\beta_{j_k}}\cdot\prod_{t=1}^{n}d_{l_t}^{\beta_{l_t}}$ be the prime factorization of x and y, respectively, where $p_{i_h}\neq d_{l_t}, q_{j_k}\neq d_{l_t}, p_{i_h}\neq q_{j_k}, \forall h=\overline{1,r}, k=\overline{1,s}, t=\overline{1,n}$. Of course $x\cdot y=\prod_{h=1}^{r}p_{i_h}^{\alpha_{i_h}}\cdot\prod_{k=1}^{s}q_{j_k}^{\beta_{j_k}}\cdot\prod_{t=1}^{n}d_{l_t}^{\alpha_{l_t}+\beta_{l_t}}$, so $g(x\cdot y)=\prod_{h=1}^{r}p_{i_h}^{\overline{3-\alpha_{i_h}}}\cdot\prod_{k=1}^{s}q_{j_k}^{\overline{3-\beta_{j_k}}}\cdot\prod_{t=1}^{n}d_{l_t}^{\overline{3-(\alpha_{l_t}+\beta_{l_t})}}$. On the other hand, $g(x)=\prod_{h=1}^{r}p_{i_h}^{\overline{3-\overline{\alpha_{i_h}}}}\cdot\prod_{t=1}^{n}d_{l_t}^{\overline{3-\overline{\alpha_{l_t}}}}$ and $g(y)=\prod_{k=1}^{s}q_{j_k}^{\overline{3-\overline{\beta_{j_k}}}}\cdot\prod_{t=1}^{n}d_{l_t}^{\overline{3-\overline{\beta_{l_t}}}}$, so that

$$g^2(g(x)\cdot g(y))=g^2\Big(\prod_{h=1}^{r}p_{i_h}^{\overline{3-\overline{\alpha_{i_h}}}}\cdot\prod_{k=1}^{s}q_{j_k}^{\overline{3-\overline{\beta_{j_k}}}}\cdot\prod_{t=1}^{n}d_{l_t}^{\overline{3-\overline{\alpha_{l_t}}}+\overline{3-\overline{\beta_{l_t}}}}\Big)=\prod_{h=1}^{r}p_{i_h}^{\overline{3-\overline{3-\overline{3-\overline{\alpha_{i_h}}}}}}\cdot\prod_{k=1}^{s}q_{j_k}^{\overline{3-\overline{3-\overline{3-\overline{\beta_{j_k}}}}}}\cdot$$

$$\cdot\prod_{t=1}^{n}d_{l_t}^{\overline{3-\overline{3-(\overline{3-\overline{\alpha_{l_t}}}+\overline{3-\overline{\beta_{l_t}}})}}}=\prod_{h=1}^{r}p_{i_h}^{\overline{3-\overline{\alpha_{i_h}}}}\cdot\prod_{k=1}^{s}q_{j_k}^{\overline{3-\overline{\beta_{j_k}}}}\cdot\prod_{t=1}^{n}d_{l_t}^{\overline{3-(\alpha_{l_t}+\beta_{l_t})}}=g(x\cdot y),$$ because

$\overline{3-\overline{3-\overline{3-\bar{a}}}}=\overline{3-\bar{a}}$ and $\overline{3-\overline{3-(\overline{3-\bar{a}}+\overline{3-\bar{b}})}}=\overline{3-\overline{(a+b)}}, \forall a,b\in\mathbf{N}$.

REMARK 8. *In the case when* $(x,y)=1$ *we obtain more simply the same result. Because* $(x,y)=1\Rightarrow (g(x),g(y))=1\Rightarrow(g^2(x),g^2(y))=1$ *so we have:*

$$g^2(g(x)\cdot g(y))=g(g(g(x)\cdot g(y)))=g(g(g(x))\cdot g(g(y)))=g(g^2(x)\cdot g^2(y))=$$
$$=g(g^2(x))\cdot g(g^2(y))=g^3(x)\cdot g^3(y)=g(x)\cdot g(y)=g(x\cdot y).$$

REMARK 9. *If* $(x,y)=1$, *then* $g(xyz)=g^2(g(xy)\cdot g(z))=g^2(g(x)g(y)g(z))$ *and this property can be extended for a finite number of factors, therefore if* $(x_1,x_2)=(x_2,x_3)=\cdots=(x_{n-2},x_{n-1})=1$, *then* $g(\prod_{i=1}^{n}x_i)=g^2(\prod_{i=1}^{n}g(x_i))$.

PROPERTY 9. *The function g has not fixed points* $x\neq 1$.

Proof. We must prove that the equation $g(x)=x$ has not solutions x>1.

Let $x=p_{i_1}^{\alpha_{i_1}}\cdot p_{i_2}^{\alpha_{i_2}}\cdot\ldots\cdot p_{i_r}^{\alpha_{i_r}}, \alpha_{i_j}\geq 1, j=\overline{1,r}$ be the prime factorization of x. Then $g(x)=\prod_{j=1}^{r}p_{i_j}^{\overline{3-\overline{\alpha_{i_j}}}}$ implies that $\alpha_{i_j}=\overline{3-\overline{\alpha_{i_j}}}, \forall j\in\overline{1,r}$ which is not possible.

REMARK 10. *The function* $\tilde{g}$ *has fixed points only in the case* $m=2k$, $k\in N^*$. *These points are* $x=p_{i_1}^{k}\cdot p_{i_2}^{k}\cdot\ldots\cdot p_{i_r}^{k}$, *where* $p_{i_j}, j=\overline{1,r}$ *are prime numbers.*

PROPERTY 10. *If* $\left(\frac{x}{(x,y)}, y\right) = 1$ *and* $\left(\frac{y}{(x,y)}, x\right) = 1$ *then we have* $g((x,y)) = (g(x), g(y))$, *where we denote by* (x,y) *the greatest common divisor of* x *and* y.

Proof. Because $\left(\frac{x}{(x,y)}, y\right) = 1$ and $\left(\frac{y}{(x,y)}, x\right) = 1$, we have $\left(\frac{x}{(x,y)}, (x,y)\right) = 1$ *and* $\left(\frac{y}{(x,y)}, (x,y)\right) = 1$, then x and y have the following prime factorization: $x = \prod_{h=1}^{r} p_{i_h}^{\alpha_{i_h}} \cdot \prod_{t=1}^{n} d_{l_t}^{\alpha_{l_t}}$ and $y = \prod_{k=1}^{s} q_{j_k}^{\beta_{j_k}} \cdot \prod_{t=1}^{n} d_{l_t}^{\alpha_{l_t}}$, $p_{i_h} \neq d_{l_t}, q_{j_k} \neq d_{l_t}$, $p_{i_h} \neq q_{j_k}$; $\forall h = \overline{1,r}, k = \overline{1,s}, t = \overline{1,n}$. Then $(x,y) = \prod_{t=1}^{n} d_{l_t}^{\alpha_{l_t}}$, therefore $g((x,y)) = \prod_{t=1}^{n} d_{l_t}^{\overline{\overline{3-\alpha_{l_t}}}}$. On the other hand

$$(g(x), g(y)) = \left(\prod_{h=1}^{r} p_{i_h}^{\overline{\overline{3-\alpha_{i_h}}}} \cdot \prod_{t=1}^{n} d_{l_t}^{\overline{\overline{3-\alpha_{l_t}}}}, \prod_{k=1}^{s} q_{j_k}^{\overline{\overline{3-\beta_{j_k}}}} \cdot \prod_{t=1}^{n} d_{l_t}^{\overline{\overline{3-\alpha_{l_t}}}}\right) = \prod_{t=1}^{n} d_{l_t}^{\overline{\overline{3-\alpha_{l_t}}}}$$

and the assertion follows.

REMARK 11. *In the same conditions,* $\tilde{g}((x,y)) = (\tilde{g}(x), \tilde{g}(y)), \forall x,y \in N^*$.

PROPERTY 11. *If* $\left(\frac{x}{(x,y)}, y\right) = 1$ *and* $\left(\frac{y}{(x,y)}, x\right) = 1$ *then we have:* $g([x,y]) = [g(x), g(y)]$, *where* (x,y) *has the above significance and* $[x,y]$ *is the least common multiple of* x *and* y.

Proof. We have the prime factorization of x and y used in the proof of the above property, therefore:

$$g([x \cdot y]) = g\left(\prod_{h=1}^{r} p_{i_h}^{\alpha_{i_h}} \cdot \prod_{k=1}^{s} q_{j_k}^{\beta_{j_k}} \cdot \prod_{t=1}^{n} d_{l_t}^{\alpha_{l_t}}\right) = \prod_{h=1}^{r} p_{i_h}^{\overline{\overline{3-\alpha_{i_h}}}} \cdot \prod_{k=1}^{s} q_{j_k}^{\overline{\overline{3-\beta_{j_k}}}} \cdot \prod_{t=1}^{n} d_{l_t}^{\overline{\overline{3-\alpha_{l_t}}}}$$

and

$$[g(x), g(y)] = \left[\prod_{h=1}^{r} p_{i_h}^{\overline{\overline{3-\alpha_{i_h}}}} \cdot \prod_{t=1}^{n} d_{l_t}^{\overline{\overline{3-\alpha_{l_t}}}}, \prod_{k=1}^{s} q_{j_k}^{\overline{\overline{3-\beta_{j_k}}}} \cdot \prod_{t=1}^{n} d_{l_t}^{\overline{\overline{3-\alpha_{l_t}}}}\right] =$$

$$= \prod_{h=1}^{r} p_{i_h}^{\overline{\overline{3-\alpha_{i_h}}}} \cdot \prod_{k=1}^{s} q_{j_k}^{\overline{\overline{3-\beta_{j_k}}}} \cdot \prod_{t=1}^{n} d_{l_t}^{\overline{\overline{3-\alpha_{l_t}}}},$$

so we have $g([x,y]) = [g(x), g(y)]$.

REMARK 12. *In the same conditions,* $\tilde{g}([x,y]) = [\tilde{g}(x), \tilde{g}(y)], \forall x,y \in N^*$.

CONSEQUENCE 4. *If* $\left(\frac{x}{(x,y)}, y\right) = 1$ *and* $\left(\frac{y}{(x,y)}, x\right) = 1$, *then* $g(x) \cdot g(y) = g((x,y)) \cdot g([x,y])$ *for every* $x,y \in N^*$.

Proof. Because $[x,y]=\frac{xy}{(x,y)}$ we have $[g(x),g(y)]=\frac{g(x)\cdot g(y)}{(g(x),g(y))}$ and using the last two properties we have:

$$g(x)\cdot g(y)=(g(x),g(y))\cdot[g(x),g(y)]=g((x,y))\cdot g([x,y]).$$

REMARK 13. *In the same conditions, we also have* $\tilde{g}(x)\cdot\tilde{g}(y)=\tilde{g}((x,y))\cdot\tilde{g}([x,y])$ *for every* $x,y\in N^{*}$.

PROPERTY 13. *The sumatory numerical function of the function g is*

$$F(n)=\prod_{j=1}^{k}\left(\frac{\alpha_{i_j}+\overline{\overline{3-\alpha_{i_j}}}}{3}(1+p_{i_j}+p_{i_j}^{2})+h_{p_{i_j}}(\alpha_{i_j})\right),$$

where $n=p_{i_1}^{\alpha_{i_1}}\cdot p_{i_2}^{\alpha_{i_2}}\cdot\ldots\cdot p_{i_k}^{\alpha_{i_k}}$ *is the prime factorization of n, and* $h_p:N\to N$ *is the numerical function defined by* $h_p(\alpha)=\begin{cases}1 & \text{for } \alpha=3k\\ -p & \text{for } \alpha=3k+1\\ 0 & \text{for } \alpha=3k+2\end{cases}$, *where p is a given number.*

Proof. Because the sumatory function of g is defined as $F(n)=\sum_{d/n}g(d)$ and because $(p_{i_1}^{\alpha_{i_1}},\prod_{t=2}^{k}p_{i_t}^{\alpha_{i_t}})=1$ and g is a multiplicative function, we have:

$$F(n)=\left(\sum_{d_1/p_{i_1}^{\alpha_1}}g(d_1)\right)\cdot\left(\sum_{d_2/p_{i_2}^{\alpha_2}\ldots p_{i_k}^{\alpha_k}}g(d_2)\right)$$

and so on, making a finite number of steps we obtain: $F(n)=\prod_{j=1}^{k}F(p_{i_j}^{\alpha_{i_j}})$.

But it is easy to prove that:

$$F(p^{\alpha})=\begin{cases}\frac{\alpha}{3}(1+p+p^2)+1 & \text{for } \alpha=3k;\\ \frac{\alpha+2}{3}(1+p+p^2)-p & \text{for } \alpha=3k+1;\\ \frac{\alpha+1}{3}(1+p+p^2) & \text{for } \alpha=3k,\ k\in\mathbf{N}, \text{ for every prime } p\end{cases}$$

Using the function h_p, we can write $F(p^{\alpha})=\frac{\overline{\overline{3-\alpha}}}{3}(1+p+p^2)+h_p(\alpha)$, therefore we have the demanded expresion of $F(n)$.

REMARK 14. *The expresion of F(n), where F is the sumatory function of* $\tilde{g}$, *is similary, but it is necessary to replace*

$\frac{\alpha_{i_j}+3-\overline{\alpha_{i_j}}}{3}$ *by* $\frac{\alpha_{i_j}+m-\overline{\alpha_{i_j}}}{m}$ *(where* $\overline{\alpha_{i_j}}$ *is now the remainder of the division of* α_{i_j} *by m and the sum* $1+p_{i_j}+p_{i_j}^2$ *by* $\sum_{k=0}^{m-1} p_{i_j}^k$*) and to define an adapted function* h_p.

In the sequel we study some equations which involve the function g.

1. *Find the solutions of the equations* $x \cdot g(x) = a$, *where* $x, a \in N^*$.

If a is not a perfect cube, then the above equation has not solutions.

If a is a perfect cube, $a = b^3, b \in N^*$, where $b = p_{i_1}^{\alpha_{i_1}} \cdot p_{i_2}^{\alpha_{i_2}} \cdot \ldots \cdot p_{i_k}^{\alpha_{i_k}}$ is the prime factorization of b, then, taking into account of the definition of the function g, we have the solutions $x = b^3 / d_{i_1 i_2 \ldots i_k}$ where $d_{i_1 i_2 \ldots i_k}$ can be every product $p_{i_1}^{\beta_1} p_{i_2}^{\beta_2} \cdots p_{i_k}^{\beta_k}$ where $\beta_1, \beta_2, \ldots, \beta_k$ take an arbitrary value which belongs of the set $\{0, 1, 2\}$.

In the case when $\beta_1 = \beta_2 = \cdots = \beta_k = 0$ we find the special solution $x = b^3$, when $\beta_1 = \beta_2 = \cdots = \beta_k = 1$, the solution $p_{i_1}^{3\beta_1 - 1} p_{i_2}^{3\beta_2 - 1} \cdots p_{i_k}^{3\beta_k - 1}$ and when $\beta_1 = \beta_2 = \cdots = \beta_k = 2$, the solution $p_{i_1}^{3\beta_1 - 2} p_{i_2}^{3\beta_2 - 2} \cdots p_{i_k}^{3\beta_k - 2}$.

We find in this way $1 + 2C_k^1 + 2^2 C_k^1 + \cdots + 2^k C_k^k = 3^k$ different solutions, where k is the number of the prime divisors of b.

2. *Prove that the following equations have not natural solutions:*

$xg(x) + yg(y) + zg(z) = 4$ *or* $xg(x) + yg(y) + zg(z) = 5$. *Give a generalization.*

Because $xg(x) = a^3, yg(y) = b^3, zg(z) = c^3$ and the equations $a^3 + b^3 + c^3 = 4$ or $a^3 + b^3 + c^3 = 5$ have not natural solutions, then the assertion holds.

We can also say thet the equations $(xg(x))^n + (yg(y))^n + (zg(z))^n = 4$ or $(xg(x))^n + (yg(y))^n + (zg(z))^n = 5$ have not natural solutions, because the equations $a^{3n} + b^{3n} + c^{3n} = 4$ or $a^{3n} + b^{3n} + c^{3n} = 5$ have not.

3. *Find all solutions of the equation* $xg(x) - yg(y) = 999$.

Because $xg(x) = a^3$ and $yg(y) = b^3$ we must give the solutions of the equation $a^3 - b^3 = 999$, wh²ich are (a=10, b=1) and (a=12,b=9).

In the first case: a=10, b=1 we have $xa(x) = 10^3 = 2^3 \cdot 5^3$

$$\Rightarrow x_0 \in \{10^3, 2^2 \cdot 5^3, 2^3 \cdot 5^2, 2 \cdot 5^3, 2^3 \cdot 5, 2^2 \cdot 5^2, 2^2 \cdot 5, 2 \cdot 5^2, 2 \cdot 5\}$$

and yb(y)=1 $\Rightarrow y_0 = 1$ so we have 9 different solutions (x_0, y_0).

In the second case: a=12, b=9 we have $xa(x) = 12^3 = 2^6 \cdot 3^3$

$$\Rightarrow x_0 \in \{2^6 \cdot 3^3, 2^5 \cdot 3^3, 2^6 \cdot 3^2, 2^4 \cdot 3^3, 2^6 \cdot 3, 2^5 \cdot 3^2, 2^4 \cdot 3^2, 2^5 \cdot 3, 2^4 \cdot 3\}$$

and yb(y)=$9^3 = 3^9 \Rightarrow y_0 \in \{3^9, 3^8, 3^7\}$ so we have another $9 \cdot 3 = 27$ different solutions (x_0, y_0).

4. *It is easy to observe that the equation g(x)=1 has an infinite number of solutions: all perfect cube numbers.*

5. *Find the solutions of the of the equation* $g(x)+g(y)+g(z)=g(x)g(y)g(z)$.

The same problem when the function is $\tilde{g}$.

It is easy to prove that the solutions are, in the first case, the permutations of the sets $\{u^3, 4v^3, 9t^3\}$, where $u, v, t \in N^*$, and in the second case $\{u^m, 2^{m-1}v^m, 3^{m-1}t^m\}$, $u, v, t \in N^*$.

Using the same ideea of [1], it is easy to find the solutions of the following equations which involve the function g:

a) $g(x)=kg(y)$, $k \in N^*, k>1$

b) $Ag(x)+Bg(y)+Cg(z)=0$, $A, B, C \in Z^*$

c) $Ag(x)+Bg(y)=C$, $A, B, C \in Z^*$, and to find also the solutions of the above equations when we replace the function g by $\tilde{g}$.

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Current Address: University of Craiova, Department of Mathematics, 13, "A.I.Cuza" street, Craiova-1100, ROMANIA