All Solutions of the Equation S(n) + d(n) = n

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The number of divisors function d(n), is a classic function of number theory, having been defined centuries ago. In contrast, the Smarandache function S(n), was defined only a few decades ago. The purpose of this paper is to find all solutions to a simple equation involving both functions.

Theorem: The only solutions to the equation

$$S(n) + d(n) = n, \quad n > 0$$

are 1, 8 and 9.

Proof: Since S(1) = 0 and d(1) = 1 we have verified the special case of n = 1.

Furthermore, with S(p) = p for p a prime, it follows that any solution must be composite.

The following results are well-known.

a) $d(p_1^{a1} \ldots p_k^{ak}) = (a1 + 1) \ldots (ak + 1)$
b) $S(p^k) \leq kp$
c) $S(p_1^{a1} \ldots p_k^{ak}) = \max \{ S(p_1^{a1}) \ldots S(p_k^{ak}) \}$

Examining the first few powers of 2.

$S(2^2) = 4, \quad d(2^2) = 3$
$S(2^3) = 4$ and $d(2^3) = 4$ which is a solution.
$S(2^4) = 6, \; d(2^4) = 5$

and in general

$$S(2^k) \leq 2k \quad \text{and} \quad d(2^k) = k + 1.$$

It is an easy matter to verify that

$$2k + k + 1 = 3k + 1 < 2^k$$

for $k > 4$.

Examining the first few powers of 3

$S(3^2) = 6$ and $d(3^2) = 3$, which is a solution.
$S(3^3) = 9$, $d(3^3) = 4$

and in general, $S(3^k) \leq 3k$ and $d(3^k) = k + 1$.

It is again an easy matter to verify that

$$3k + k + 1 < 3^k$$

for $k > 3$.

Consider $n = p^k$ where $p > 3$ is prime and $k > 1$. The expression becomes

$S(p^k) + d(p^k) \leq kp + k + 1 = k(p+1) + 1$.

Once again, it is easy to verify that this is less than p^k for $p \geq 5$.

Now, assume that $n = p_1^{a1} \ldots p_k^{ak}$, $k > 1$ is the unique prime factorization of n.

Case 1: $n = p_1 p_2$, where $p_2 > p_1$. Then $S(n) = p_2$ and $d(n) = 2 * 2 = 4$. Forming the sum,

$$p_2 + 4$$

we then examine the subcases.

Subcase 1: $p_1 = 2$. The first few cases are

$n = 2 * 3$, $S(n) + d(n) = 7$
$n = 2 * 5$, $S(n) + d(n) = 9$
$n = 2 * 7$, $S(n) + d(n) = 11$
$n = 2 * 11$, $S(n) + d(n) = 15$

and it is easy to verify that $S(n) + d(n) < n$, for p_2 a prime greater than 11.

Subcase 2: $p_1 = 3$. The first few cases are

$n = 3 * 5$, $S(n) + d(n) = 5 + 4$
$n = 3 * 7$, $S(n) + d(n) = 7 + 4$
$n = 3 * 11$, $S(n) + d(n) = 11 + 4$

and it is easy to verify that $S(n) + d(n) < n$ for p_2 a prime greater than 11.

Subcase 3: It is easy to verify that

$$p_2 + 4 < p_1p_2$$

for $p_1 \geq 5, p_2 > p_1$.

Therefore, there are no solutions for $n = p_1p_2, \ p_1 < p_2$.

Case 2: $n = p_1p_2^{a_2}$, where $a_2 > 1$ and $p_1 < p_2$. Then $S(n) \leq a_2p_2$ and $d(n) = 2(a_2 + 1)$.

$S(n) + d(n) \leq a_2p_2 + 2(a_2 + 1) = a_2p_2 + 2a_2 + 2$

We now induct on a_2 to prove the general inequality

$$a_2p_2 + 2a_2 + 2 < p_1p_2^{a_2}$$

Basis step: $a_2 = 2$. The formula becomes

$2p_2 + 4 + 2 = 2p_2 + 6$ on the left and

$p_1p_2p_2$ on the right. Since $p_2 \geq 3$, $2 + \frac{6}{p_2} \leq 4$ and $p_1p_2 \geq 6$. Therefore,

$$2 + \frac{6}{p_2} < p_1p_2$$

and if we multiply everything by p_2, we have

$$2p_2 + 6 < p_1p_2p_2.$$

Inductive step: Assume that the inequality is true for $k \geq 2$

$$kp_2 + 2k + 2 < p_1p_2^k.$$

and examine the case where the exponent is $k + 1$.

$$(k + 1)p_2 + 2(k + 1) + 2 = kp_2 + p_2 + 2k + 2 + 2 = (kp_2 + 2k + 2) + p_2 + 2$$

$$< p_1p_2^k + p_2 + 2 \qquad \text{by the inductive hypothesis.}$$

Since $p_1p_2^k$ when $k \geq 2$ is greater than $p_2 + 2$ is follows that

$$p_1p_2^k + p_2 + 2 < p_1p_2^{k+1}.$$

Therefore, $S(n) + d(n) < n$, where $n = p_1p_2^k$, $k \geq 2$.

Case 3: $n = p_1^{a_1} p_2$, where $a_1 > 1$.

We have two subcases for the value of S(n), depending on the circumstances

Subcase 1: $S(n) \leq a_1 p_1$.

Subcase 2: $S(n) = p_2$.

In all cases, $d(n) = 2(a_1 + 1)$.

Subcase 1: $S(n) + d(n) \leq a_1 p_1 + 2(a_1 + 1) = a_1 p_1 + 2a_1 + 2$.

Using an induction argument very similar to that applied in case 2, it is easy to prove that the inequality

$$a_1 p_1 + 2a_1 + 2 < p_1^{a_1} p_2.$$

is true for all $a_1 \geq 2$.

Subcase 2: $S(n) + d(n) = p_2 + 2(a_1 + 1) = p_2 + 2a_1 + 2$

It is again a simple matter to verify that the inequality

$$p_2 + 2a_1 + 2 < p_1^{a_1} p_2$$

is true for all $a_1 \geq 2$.

Case 4: $n = p_1^{a_1} p_2^{a_2}$, where $p_1 < p_2$ and $a_1, a_2 \geq 2$.

$d(n) = (a_1 + 1)(a_2 + 1)$

Subcase 1: $S(n) \leq a_1 p_1$

$$S(n) + d(n) \leq a_1 p_1 + (a_1 + 1)(a_2 + 1) < p_1^{a_1} + p_1^{a_1}(a_2 + 1) = p_1^{a_1}(a_2 + 2) < p_1^{a_1} p_2^{a_2}$$

Subcase 2: $S(n) \leq a_2 p_2$

$$S(n) + d(n) \leq a_2 p_2 + (a_1 + 1)(a_2 + 1) < p_2^{a_2} + p_2^{a_2}(a_1 + 1) = p_2^{a_2}(a_1 + 2) < p_1^{a_1} p_2^{a_2}$$

Case 5: $n = p_1^{a_1} \ldots p_k^{a_k}$, where $k \geq 2$.

The proof is by induction on k.

Basis step: Completed in the first four cases.

Inductive step: Assume that for $n_1 = p_1^{a_k} \ldots p_k^{a_k}$, $k \geq 2$

$$a_i p_i + (a_1 + 1) \ldots (a_k + 1) < n_1$$

where $S(n_1) \leq a_i p_i$. Which means that

$$S(n_1) + d(n_1) < n_1.$$

Consider $n_2 = p_1^{a_1} \ldots p_k^{a_k} p_{k+1}^{a_{k+1}}$.

Subcase 1: $S(n_2) = S(n_1)$. Since $p_{k+1} \geq 5$, it follows that $(a_{k+1} + 1) < p_{k+1}^{a_{k+1}}$ and we can this in combination with the inductive hypothesis to conclude

$$a_i p_i + (a_1 + 1) \ldots (a_k + 1)(a_{k+1} + 1) < n_1 p_{k+1}^{a_{k+1}},$$

which implies that $S(n_2) + d(n_2) < n_2$.

Subcase 2: $S(n_2) > S(n_1)$, which implies that $S(n_2) \leq a_{k+1} p_{k+1}$. Starting with the inductive hypotheses

$$a_i p_i + (a_1 + 1) \ldots (a_k + 1) < p_1^{a_1} \ldots p_k^{a_k}$$

and multply both sides by $a_{k+1} p_{k+1}$ to obtain the inequality

$$a_i p_i a_{k+1} p_{k+1} + a_{k+1} p_{k+1} (a_1 + 1) \ldots (a_k + 1) < p_1^{a_1} \ldots p_k^{a_k} a_{k+1} p_{k+1}$$

Since $p_{k+1} \geq 5$, it follows that

$$p_1^{a_1} \ldots p_k^{a_k} a_{k+1} p_{k+1} \leq p_1^{a_1} \ldots p_k^{a_k} p_{k+1}^{a_{k+1}}$$

and with $a_{k+1} p_{k+1} > (a_{k+1} + 1)$, we have

$$a_{k+1} p_{k+1} + (a_1 + 1) \ldots (a_k + 1)(a_{k+1} + 1) <$$
$$a_i p_i a_{k+1} p_{k+1} + a_{k+1} p_{k+1} (a_1 + 1) \ldots (a_k + 1).$$

Combining the inequalities, we have

$$a_{k+1} p_{k+1} + (a_1 + 1) \ldots (a_k + 1)(a_{k+1} + 1) < p_1^{a_1} \ldots p_k^{a_k} p_{k+1}^{a_{k+1}}$$

which implies

$$S(n_2) + d(n_2) < n.$$

Therefore, the only solutions to the equation

$$S(n) + d(n) = n$$

are 1, 8 and 9.