AN ASYMPTOTIC FORMULA INVOLVING SQUARE COMPLEMENT NUMBERS

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ABSTRACT. The main purpose of this paper is to study the mean value properties of the square complement number sequence $\{S(n)\}$, and give an interesting asymptotic formula involving $S(n)$.

1. INTRODUCTION AND RESULTS

For each positive integer n, we call $S(n)$ as a square complement number of n, if $S(n)$ is the smallest positive integer such that $nS(n)$ is a perfect square. In reference [1], Professor F.Smarandache asked us to study the properties of the sequence $\{S(n)\}$. About this problem, we know very little at present. The main purpose of this paper is to study the asymptotic property of this sequence, and obtain an interesting asymptotic formula involving square complement numbers. That is, we shall prove the following result:

Theorem. *Let real number $x \geq 3$, $S(n)$ denotes the square complement number of n. Then we have the asymptotic formula*

$$\sum_{n\leq x} d(S(n)) = c_1 x \ln x + c_2 x + O(x^{\frac{1}{2}+\epsilon}),$$

where $d(n)$ is the divisor function, $\epsilon > 0$ be any fixed real number, c_1 and c_2 are defined as following:

$$c_1 = \frac{6}{\pi^2}\prod_p \left(1 - \frac{1}{(p+1)^2}\right),$$

$$c_2 = \frac{6}{\pi^2}\prod_p \left(1 - \frac{1}{(p+1)^2}\right)\left(\sum_p \frac{2(2p+1)\ln p}{(p-1)(p+1)(p+2)} + 2\gamma - 1\right),$$

the product and summation over all prime p, γ is the Euler's constant.

Key words and phrases. Square complement numbers; Sequence; Asymptotic formula.

2. Proof of the Theorem

In this section, we shall complete the proof of the Theorem. First we need the following:

Lemma. *Let real number $y \geq 3$, then we have the asymptotic formula*

$$\sum_{n\leq y} d(n)|\mu(n)| = c_1' y \ln y + c_2' y + O(y^{\frac{1}{2}+\epsilon}),$$

where $\mu(n)$ is the Möbius function, c_1' and c_2' are defined as following:

$$c_1' = \frac{36}{\pi^4}\prod_p \left(1 - \frac{1}{(p+1)^2}\right),$$

$$c_2' = \frac{36}{\pi^4}\prod_p \left(1 - \frac{1}{(p+1)^2}\right)\left(\sum_p \frac{2\ln p}{(p+1)(p+2)} + \sum_p \frac{4\ln p}{p^2-1} + 2\gamma - 1\right)$$

Proof. Let $T = \sqrt{y}$, $A(s) = \prod_p \left(1 - \frac{1}{(p^s+1)^2}\right)$. Then from the Perron formula (See Theorem 2 of reference [2]) , we can obtain

$$\sum_{n\leq y} d(n)|\mu(n)| = \frac{1}{2\pi i}\int_{1+\epsilon-iT}^{1+\epsilon+iT} \frac{\zeta^2(s)}{\zeta^2(2s)} A(s)\frac{y^s}{s} ds + O(y^{\frac{1}{2}+\epsilon}),$$

where $\mu(n)$ is the Möbius function, $\epsilon > 0$ be any real number.

Moving the integeration line to $\mathrm{Re}(s) = \frac{1}{2} + \epsilon$, here $s = 1$ is a second order pole of $\frac{\zeta^2(s)}{\zeta^2(2s)} A(s)\frac{y^s}{s}$, and the residue of this function at $s = 1$ is

$$\operatorname*{Res}_{s=1}\left(\frac{\zeta^2(s)}{\zeta^2(2s)} A(s)\frac{y^s}{s}\right) = c_1' y \ln y + c_2' y.$$

where c_1' and c_2' are defined as following:

$$c_1' = \frac{36}{\pi^4}\prod_p \left(1 - \frac{1}{(p+1)^2}\right),$$

$$c_2' = \frac{36}{\pi^4}\prod_p \left(1 - \frac{1}{(p+1)^2}\right)\left(\sum_p \frac{2\ln p}{(p+1)(p+2)} + \sum_p \frac{4\ln p}{p^2-1} + 2\gamma - 1\right).$$

Hence,

$$\sum_{n\leq y} d(n)|\mu(n)| = c_1' y \ln y + c_2' y + O(y^{\frac{1}{2}+\epsilon})$$

This proves the Lemma.

Now, we shall complete the proof of the Theorem. From the above Lemma we have

$$\begin{aligned}
\sum_{n\le x} d(S(n)) &= \sum_{ak^2\le x} d(S(ak^2)) \\
&= \sum_{ak^2\le x} d(a)|\mu(a)| \\
&= \sum_{k\le\sqrt{x}} \sum_{a\le\frac{x}{k^2}} d(a)|\mu(a)| \\
&= \sum_{k\le\sqrt{x}} \left(c_1' \frac{x}{k^2}\ln\frac{x}{k^2} + c_2'\frac{x}{k^2} + O\left(\frac{x^{\frac{1}{2}+\epsilon}}{k^{1+2\epsilon}}\right)\right) \\
&= c_1'\zeta(2)x\ln x + (c_2'\zeta(2) + 2c_1'\zeta'(2))x + O\left(x^{\frac{1}{2}+\epsilon}\right). \qquad (1)
\end{aligned}$$

Let

$$c_1 = c_1'\zeta(2) = \frac{6}{\pi^2}\prod_p\left(1-\frac{1}{(p+1)^2}\right), \qquad (2)$$

and

$$\begin{aligned}
c_2 &= c_2'\zeta(2) + 2c_1'\zeta'(2) = c_2'\zeta(2) - 2c_1'\zeta(2)\sum_p \frac{\ln p}{p^2-1} \\
&= \frac{6}{\pi^2}\prod_p\left(1-\frac{1}{(p+1)^2}\right)\left(\sum_p \frac{2(2p+1)\ln p}{(p-1)(p+1)(p+2)} + 2\gamma - 1\right). \qquad (3)
\end{aligned}$$

Combining (1), (2) and (3), we immediately deduce the asymptotic formula

$$\sum_{n\le x} d(S(n)) = c_1 x\ln x + c_2 x + O(x^{\frac{1}{2}+\epsilon}),$$

This completes the proof of the Theorem.

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References

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